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Investigation of Tutte polynomial of Graphs

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Investigation of Tutte polynomial of Graphs

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Referat

The Tutte polynomial is an important tool in graph theory. This paper provides an introduction to the two-variable polynomial using the spanning subgraph and rank-generating polynomials. The equivalency of definitions is shown in detail, as well as evaluations and derivatives. The properties and examples of the polynomial, i.e. the universality, coefficient relations, closed forms and recurrence relations are mentioned. Moreover, the thesis contains the connection between the dichromate and other significant polynomials.

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1 Introduction

1.1 Structure

The paper begins with a preface into the history of the Tutte polynomial of graphs. In the first chapter we introduce notions crucial for the comprehension of the topic.
The second chapter focuses on the definitions of the polynomial with a preamble for the spanning subgraph and Whitney rank-generating polynomial. We show the proof of equivalency of different definitions and provide a detailed example of calculating the different polynomials.
Chapter three mentions evaluations of the polynomial at precise points. These evaluations give information about the graph's features: spanning subgraphs, spanning forests, orientations, etc.
The next chapter specializes in the properties of the polynomial regarding planar graphs and their duals, components of a graph, as well as concepts for the coefficients of the spanning forest expansion formula. Chapter five and six include notions and theorems about the universality of the Tutte polynomial and the rooted graphs expansion, respectively.
A brief introduction and relations of other polynomials which are specialisations of the Tutte polynomial follows in the last chapter. It includes: the chromatic, bad coloring, flow and reliability polynomials. We conclude the paper with details and links to further extensive research on the topic.

1.2 History

Ellis-Monaghan and Merino [EM11] said: 'A graph polynomial encodes information about a graph. The challenge is in extracting combinatorially useful information from this algebraic object.'
Throughout the history of graph theory, which is a new and constantly evolving branch of mathematics, graph polynomials have captured the attention of many mathematicians. W.T. Tutte, H. Whitney and H.H. Crapo are the names related to the Tutte polynomial investigated in this paper.
As will be seen later, the first use of, what is now called the *Tutte polynomial*, was by Hassler Whitney who 'knew and used analogous coefficients without bothering to affix them to two variables', said W.T. Tutte.
Inspired by perfect rectangles and spanning trees, William Thomas Tutte started working on the deletion/contraction formula early in his days at Trinity College. He was also interested in whether this recursive formula could be used for other graph invariants. Initially, he called the invariants which satisfy the relation - *W-functions* - or *V-functions*, if they were also multiplicative over the components. Investigation of these functions led him to a two-variable polynomial that could be related to the chromatic and flow polynomial using a specialization, i.e. taking one of the variables equal to zero. Although his later work on the *dichromate* polynomial - as W.T. Tutte considered it to be a generalization of the *chromatic polynomial* - includes concepts extended to matroids, the in depth generalisation on matroids is credited to Crapo [Cra69].

1.3 Preliminaries

We assume the reader has a basic knowledge of graph theory, namely that the following notions are known:

- loop, bridge, ordinary edge
- connected component, connected graph
- tree, forest, spanning tree, spanning forest
- subgraph, spanning subgraph

The focus of this paper are undirected graphs that have loops and multiple edges.
Let $G = (V, E)$ be a graph. We denote $V(G)$-the set of its vertices and $E(G)$-the set of its edges. The **size** of a graph is the number of edges, $|E|$; the **order** of a graph is the number of vertices, $|V|$. $k(E)$ denotes the number of components of the graph.

Definition 1.1
The **rank** and **co-rank(nullity)** of a graph $G = (V, E)$, denoted by $r(E)$ and $n(E)$, are defined as

$$r(E) = |V(G)| - k(E)$$

$$n(E) = |E(G)| - r(E)$$

Definition 1.2
G-e is the graph obtained from *G* by deletion/removal of the edge *e*, $e \in E(G)$; i.e. $G - e = (V, E \backslash e)$.

Definition 1.3
G/e is the graph obtained from *G* by contraction of the edge $e = \{u, v\}$, $e \in E(G)$; i.e. the graph obtained from $G - e$ by merging *u* and *v*.

Definition 1.4
Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be graphs. A **one-point join** of G and H, denoted as $G * H$, is formed by merging the vertices $u \in V_1$ and $v \in V_2$ into a single vertex c, called a *cut vertex*.

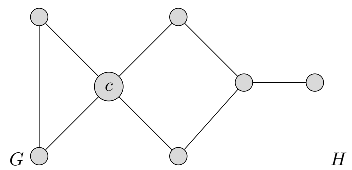

Figure 1.1: One-point join $G * H$

Definition 1.5
Let G be a graph and $E(G)$ are ordered linearly, i.e. $e_1 < e_2 < e_3 < ... < e_m$. Let $\tau(G)$ be the set of all spanning trees of G. $H \in \tau(G)$ is a spanning tree with the edge set F; $H = (V, F)$. An edge $e \in F(H)$ is **internally active** in H if e is the largest edge in the cut it defines. An edge $e \in F(H)$ is **externally active** in H if e is the largest edge in the unique cycle of $H + e$.

$Int(H)$ is the set of internally active edges of H; $int(H) = |Int(H)|$.
$Ext(H)$ is the set of externally active edges of H; $ext(H) = |Ext(H)|$.
As these notions are rather new in graph theory, there are resources that claim this definition differently, i.e. that we take the smallest and not largest edge. However, in this paper, we agree on the previously mentioned definition with the largest edge and use it throughout the chapters.

Example 1.6
To emphasize this definition, we take the bull graph as an example. We denote the graph by $B = (V, E)$ and the edges linearly ordered in the following way: $1 < 2 < 3 < 4 < 5$.

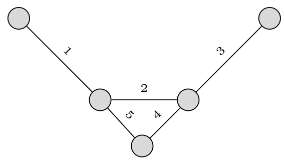

Figure 1.2: Bull graph B

The bull graph has 3 spanning trees, we take one of them, H, and analyze its internal and external activity.

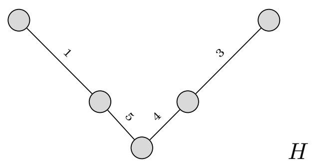

Figure 1.3: Spanning tree of the bull graph

We start with edges that are part of the tree and check for internal activity. The cut defined by 1 includes only the edge itself, thus, 1 is an internally active edge. Similarly, 3 is also internally active. Let us take edge 4: the cut defined by it in B contains both 4 and 2. However, 4 is the largest edge, meaning it is internally active. Following the same algorithm, 5 is also internally active.
Next, we move to the edges that are not part of the spanning tree, in our case only 2. In the cycle $H + 2$, 2 is not the largest edge, hence, not an externally active edge. To sum up, $Int(H) = \{1, 3, 4, 5\}$ and $Ext(H) = \emptyset$.

2 Equivalent definitions of the Tutte Polynomial

2.1 The Spanning Subgraph Polynomial

The spanning subgraph polynomial has both a recursive and a spanning forest representation and it is closely related to the Tutte polynomial. We analyse proofs of equivalency of the definitions as simpler similar proofs for the Tutte polynomial.

Definition 2.1
Let $G = (V, E)$ be a graph. The **spanning subgraph polynomial** of G is:

$$\mathcal{Z}(G; x, y) = \sum_{F \subseteq E} x^{|F|} y^{k(F)}$$

where $k(F)$ is the number of components of the graph (V, F).

Example 2.2
As an example we take the *Star graph* with 3 edges, S_3: $E(S_3) = \{a, b, c\}$.

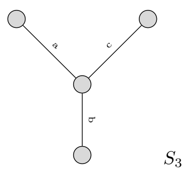

Figure 2.1: Star graph

For the calculation of the spanning subgraph polynomial, we take all possible subsets of the edge set, shown in *Figure 2.2*:

$$\mathcal{P}(E(S_3)) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}.$$

We only need the cardinality of the subsets and the number of components of the graphs defined by the edges, which yields:

$$\mathcal{Z}(S_3; x, y) = \sum_{F \subseteq E} x^{|F|} y^{k(F)} = x^{|F_0|} y^{k(F_0)} + x^{|F_{11}|} y^{k(F_{11})} + x^{|F_{12}|} y^{k(F_{12})} +$$

$$+ x^{|F_{13}|} y^{k(F_{13})} + x^{|F_{21}|} y^{k(F_{21})} + x^{|F_{22}|} y^{k(F_{22})} + x^{|F_{23}|} y^{k(F_{23})} + x^{|F_3|} y^{k(F_3)} =$$

$$= x^4 y^0 + x^1 y^2 + x^1 y^2 + x^1 y^2 + x^2 y^2 + x^2 y^2 + x^2 y^2 + x^3 y^1 =$$

$$= y^4 + 3xy^3 + 3x^2y^2 + x^3 y = y(x + y)^3$$

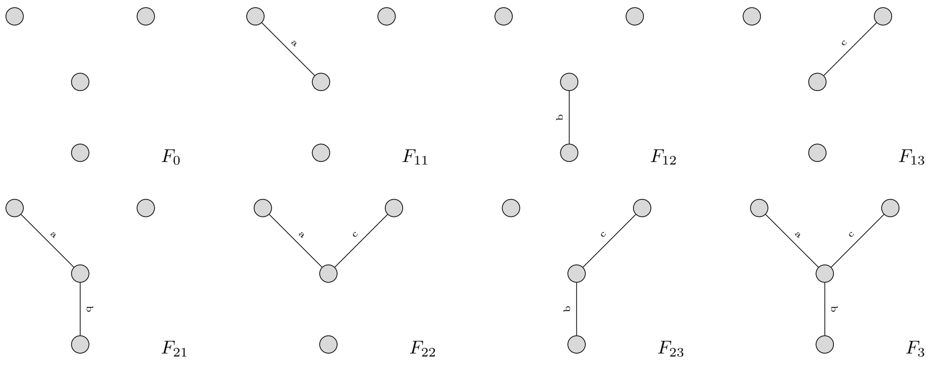

Figure 2.2: Star graph - edge subsets

Theorem 2.3
Let $G=(V,E)$ be a graph, $e \in E$. Then

$$\mathcal{Z}(G;x,y)=\mathcal{Z}(G-e;x,y)+x\mathcal{Z}(G/e;x,y)$$

Proof[Tit22]: Using the definition of the polynomial, we split it into two sums over subsets that contain e and the ones that do not:

$$\mathcal{Z}(G;x,y)=\sum_{F\subseteq E} x^{|F|}y^{k(F)}=$$

$$=\sum_{F\subseteq E\setminus e} x^{|F|}y^{k(F)}+\sum_{F:e\in F\subseteq E} x^{|F|}y^{k(F)}=$$

$$=\mathcal{Z}(G-e;x,y)+x\sum_{F\subseteq E\setminus e} x^{|F|}y^{k(F\cup e)}=$$

$$=\mathcal{Z}(G-e;x,y)+x\mathcal{Z}(G/e;x,y)$$

□

Using the same method, we conclude two more theorems.

Theorem 2.4
Let $G=(V,E)$ be a graph and $e \in E$ be a bridge. Then

$$\mathcal{Z}(G;x,y)=(x+y)\mathcal{Z}(G/e;x,y)$$

Theorem 2.5
Let $G=(V,E)$ be a graph and $e \in E$ be a loop. Then

$$\mathcal{Z}(G;x,y)=(x+1)\mathcal{Z}(G-e;x,y)$$

Let $\tau(G)$ be the set of all spanning forests of a graph $G=(V,E)$.

Theorem 2.6
Let $G = (V, E)$ be a graph, $E(G)$ be ordered linearly, $K = (V, A)$ a spanning subgraph of G. Then there exists a unique spanning forest $L = (V, B) \in \tau(G)$ such that $K = L - Int(L) + Ext(L)$.

Proof [Tit22]: We generate the desired spanning forest by this algorithm: Firstly, we set $E_1 = \emptyset$ and $E_2 = \emptyset$; $i = 0, K_0 = K$.

- *Step 1.* We assume that there is a component $K' = (W, D)$ of K_i and a component $G' = (X, F)$ of G, $W \subset X$. If this is not the case, continue to *Step 2*. We choose the largest edge e that connects a vertex of W with a vertex of $X \setminus W$. The edge e is then included in E_1 and inserted in K_i, which gives K_{i+1}. Next, we take $i + 1$ and repeat the process, i.e. *Step 1*. We exit *Step 1* when the number of components of K_i and G are the same. As a result, all edges in E_i are bridges in the graph K_i.
- *Step 2.* Let C be a cycle. If there exists a cycle C in the graph K_i, then we choose the largest edge e of it, include it in E_2 and remove e from C. This process is repeated until there are no more cycles in the graph, which makes it a spanning forest L of G.

Consequently, as E_1 consists only of largest bridges, $E_1 = Int(H)$. $E_2 = Ext(H)$, as it consists only of largest edges that are part of a cycle. □

Theorem 2.7
Let $G = (V, E)$ be a graph. The spanning subgraph can be equivalently defined by the internal and external activity of edges:

$$\mathcal{Z}(G; x, y) = x^{|V|-k(E)} y^{k(E)} \sum_{H \in \tau(G)} \left(1 + \frac{y}{x}\right)^{int(H)} (1 + x)^{ext(H)}$$

.

Proof [Tit22]: Using the previous theorem, we can build every spanning subgraph originating from a spanning forest H by removal of an arbitrary subset of $Int(H)$ and addition of an arbitrary subset from $Ext(H)$, which yields:

$$\mathcal{Z}(G; x, y) = \sum_{H \in \tau(G)} \sum_{i=0}^{int(H)} \sum_{j=0}^{ext(H)} x^{|V|-k(E)-i+j} y^{k(E)-i}$$

$$= x^{|V|-k(E)} y^{k(E)} \sum_{H \in \tau(G)} \sum_{i=0}^{int(H)} \left(\frac{y}{x}\right)^i \sum_{j=0}^{ext(H)} x^j$$

$$= x^{|V|-k(E)} y^{k(E)} \sum_{H \in \tau(G)} \left(1 + \frac{y}{x}\right)^{int(H)} (1 + x)^{ext(H)}$$

□

Theorem 2.8
Let $G = (V, E)$ be a graph, $G_1, ..., G_n$ be its components. Then

$$\mathcal{Z}(G; x, y) = \prod_{i=1}^{n} \mathcal{Z}(G_i; x, y)$$

Proof [Tit22]: We take the components in the following way: $G_1 = (V_1, E_1), ..., G_n = (V_n, E_n)$. We consider the subsets of the edge set and partition them into n subsets. Namely, $F \subseteq E$, $\sum_{i=1}^{n} F_i = F$, which means

$$\sum_{i=1}^{n} |F_i| = F, \sum_{i=1}^{n} k(F_i) = k(F)$$

We attain

$$\mathcal{Z}(G; x, y) = \sum_{F \subseteq E} x^{|F|} y^{k(F)}$$

$$= \sum_{F_1 \subseteq E_1, ..., F_n \subseteq E_n} x^{|F_1| + ... + |F_n|} y^{k(F_1) + ... + k(F_n)} =$$

$$= \sum_{F_1 \subseteq E_1} x^{|F_1|} y^{k(F_1)} ... \sum_{F_n \subseteq E_n} x^{|F_n|} y^{k(F_n)}$$

$$= \mathcal{Z}(G_1; x, y) ... \mathcal{Z}(G_n; x, y).$$

□

Theorem 2.9
Let G and H be a graphs. Then the spanning subgraph polynomial of their one-point join is the following:

$$\mathcal{Z}(G * H; x, y) = \frac{1}{y} \mathcal{Z}(G; x, y) \mathcal{Z}(H; x, y)$$

Proof [Tit22]: Let v be the cut vertex of $G * H$. Any subset of the edge set of $G * H$ can be partitioned into $F_1 \subseteq E(G)$ and $F_2 \subseteq E(H)$.
Due to the fact that v is a cut vertex formed by merging two different vertices of G and H, respectively; in $G * H$, v is counted twice. Thus $k(F) = k(F_1) + k(F_2) - 1$. Substituting it into the definition of the polynomial, we obtain:

$$\mathcal{Z}(G * H; x, y) = \sum_{F_1 \subseteq E(G), F_2 \subseteq E(H)} x^{|F_1| + |F_2|} y^{k(F_1) + k(F_2) - 1}$$

$$= \frac{1}{y} \sum_{F_1 \subseteq E(G)} x^{|F_1|} y^{k(F_1)} \sum_{F_2 \subseteq E(H)} x^{|F_2|} y^{k(F_2)}$$

$$= \frac{1}{x} \mathcal{Z}(G; x, y) \mathcal{Z}(H, x, y).$$

□

2.2 The Whitney Rank-Generating Polynomial

Definition 2.10
Let $G = (V, E)$ be a graph. The **(Whitney) rank-generating polynomial** of G is:

$$S(G; x, y) = \sum_{F \subseteq E(G)} x^{r(E)-r(F)} y^{n(F)} = \sum_{F \subseteq E(G)} x^{k(E)-k(F)} y^{n(F)}$$

Theorem 2.11
Let $G = (V, E)$ be a graph, $e \in E$. Then:

$$S(G; x, y) = \begin{cases} (x+1)S(G-e; x, y) & \text{if e is a bridge,} \\ (y+1)S(G-e; x, y) & \text{if e is a loop,} \\ S(G-e; x, y) + S(G/e; x, y) & \text{if e is neither a bridge or a loop} \end{cases}$$

Specifically, $S(E_n; x, y) = 1$ when E_n is the empty graph with n vertices.

Proof[Bol98]: Firstly, we prove the assertion for the empty graph. As there are no edges in the graph, the only subset is the edge set itself (∅) and $k(E_n) = n$. Thus

$$S(E_n; x, y) = x^{k(E)-k(E)} y^{n(E)} = x^0 y^{0-n+n} = 1$$

We set $G' = G - e, G'' = G/e$, as well as $r' = r(G'), n' = n(G'), r'' = r(G''), n'' = n(G'')$. Furthermore, if $e \in E$ and $F \subseteq E - e$, then $r(F) = r'(F), n(F) = n'(F), r(E) - r(F \cup e) = r''(E - e) - r''(F) = r(G'') - r''(F)$.

$$r(E) = \begin{cases} r'(E-e) + 1 & \text{if e is a bridge,} \\ r'(E-e) & \text{otherwise} \end{cases}$$

This happens because deleting a bridge from a graph increases the number of its components.

$$n(F \cup e) = \begin{cases} n''(F) + 1 & \text{if e is a loop,} \\ n''(F) & \text{otherwise} \end{cases}$$

This is due to the fact that, when contracting a loop, the number of edges decreases, while the number of vertices remains the same.
Nextly, we split $S(G; x, y)$ the following way:

$$S(G; x, y) = S_0(G; x, y) + S_1(G; x, y),$$

with

$$S_0(G; x, y) = \sum_{F \subseteq E, e \notin F} x^{r(E)-r(F)} y^{n(F)}$$

and

$$S_1(G; x, y) = \sum_{F \subseteq E, e \in F} x^{r(E)-r(F)} y^{n(F)}.$$

Consequently :

$$S_0(G;x,y) = \sum_{F\subseteq E-e} x^{r(E)-r(F)}y^{n(F)}$$

$$= \begin{cases} \sum\limits_{F\subseteq E(G')} x^{r'(E-e)+1-r'(F)}y^{n'(F)} & \text{if } e \text{ is a bridge,} \\ \sum\limits_{F\subseteq E(G')} x^{r'(E-e)-r'(F)}y^{n'(F)} & \text{otherwise} \end{cases}$$

$$= \begin{cases} xS(G-e;x,y) & \text{if } e \text{ is a bridge,} \\ S(G-e;x,y) & \text{otherwise} \end{cases}$$

and

$$S_1(G;x,y) = \sum_{F\subseteq E-e} x^{r(E)-r(F\cup e)}y^{n(F\cup e)}$$

$$= \begin{cases} \sum\limits_{F\subseteq E(G'')} x^{r(G'')-r''(F)}y^{n''(F)+1} & \text{if } e \text{ is a loop,} \\ \sum\limits_{F\subseteq E(G'')} x^{r(G'')-r''(F)}y^{n''(F)} & \text{otherwise} \end{cases}$$

$$= \begin{cases} yS(G/e;x,y) & \text{if } e \text{ is a loop,} \\ S(G/e;x,y) & \text{otherwise} \end{cases}$$

On that account:

$$S(G;x,y) = \begin{cases} xS(G-e;x,y) + S(G/e;x,y) & \text{if } e \text{ is a bridge,} \\ S(G-e;x,y) + yS(G/e;x,y) & \text{if } e \text{ is a loop,} \\ S(G-e;x,y) + S(G/e;x,y) & \text{if } e \text{ is neither a bridge or a loop} \end{cases}$$

Lastly, if e is a bridge: $r''(E-e) - r''(F) = r'(E-e) - r'(F)$ and $n''(F) = n'(F)$ for all $F \subset E - e$; if e is a loop : $G/e \cong G - e$.
This gives the following expression: if e is a bridge or loop, then

$$S(G/e;x,y) = S(G-e;x,y)$$

Thus,

$$S(G;x,y) = \begin{cases} (x+1)S(G-e;x,y) & \text{if } e \text{ is a bridge,} \\ (y+1)S(G-e;x,y) & \text{if } e \text{ is a loop,} \\ S(G-e;x,y) + S(G/e;x,y) & \text{if } e \text{ is neither a bridge or a loop} \end{cases}$$

□

Remark 2.12
If a graph $G = (V, E)$ consists only of i bridges and j loops, then:

$$S(G;x,y) = (x+1)^i(y+1)^j.$$

Example 2.13
As an example, we consider again the star graph, S_3, see *Figure 2.1*. As all of the edges are bridges, we obtain:

$$S(S_3; x, y) = (x+1)^3(y+1)^0 = (x+1)^3.$$

2.3 The Tutte Polynomial

The **Tutte Polynomial** has different notations in the bibliography, i.e. $T_G = T(G) = T_G(x, y)$. In this paper, we use $T(G; x, y)$.

Definition 2.14
Let $G = (V, E)$ be a graph. The **rank/co-rank definition** of the Tutte polynomial is

$$T(G; x, y) = \sum_{F \subseteq E} (x-1)^{r(E)-r(F)}(y-1)^{n(F)}$$

$$= \sum_{F \subseteq E} (x-1)^{k(F)-k(E)}(y-1)^{k(F)+|F|-|V|}$$

This form was introduced by *William Tutte* himself.

Definition 2.15
Let $G = (V, E)$ be a graph and $\tau(G)$ be the set of all the spanning forests of G. The **spanning forest expansion** of the Tutte polynomial is

$$T(G; x, y) = \sum_{H \in \mathcal{F}(G)} x^{int(H)} y^{ext(H)}.$$

Equivalently:

$$T(G; x, y) = \sum_{i,j} t_{ij} x^i y^j,$$

where t_{ij} is the number of spanning forests with internal activity i and external activity j.

Definition 2.16
Let $G = (V, E)$ be a graph, e an ordinary edge.
The **recursive formula** of the Tutte polynomial is

$$T(G; x, y) = T(G-e; x, y) + T(G/e; x, y)$$

If G consists of i bridges and j loops, then

$$T(G; x, y) = x^i y^j.$$

We can rewrite the last definition in another way.

Remark 2.17
Let $G = (V, E)$ be a graph. Then:

$$T(G; x, y) = \begin{cases} xT(G - e; x, y) & \text{if } e \text{ is a bridge,} \\ yT(G - e; x, y) & \text{if } e \text{ is a loop,} \\ T(G - e; x, y) + T(G/e; x, y) & \text{if } e \text{ is neither a bridge or a loop.} \end{cases}$$

Example 2.18
We consider again the previously mentioned Bull graph $B = (V, E)$.
For *Definition 2.15* we find the spanning trees of the graph. The bull graph has 3 spanning trees, shown in *Figure 2.3*.

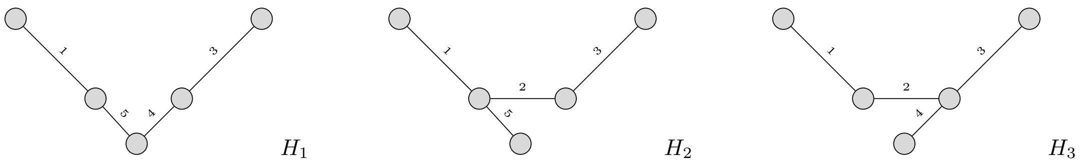

Figure 2.3: Spanning trees of the bull graph

For H_1: $Int(H_1) = \{1, 3, 4, 5\}$ and $Ext(H_1) = \emptyset$. Similarly, for H_2: $Int(H_2) = \{1, 3, 5\}$ and $Ext(H_2) = \emptyset$; and for H_3: $Int(H_3) = \{1, 3\}$ and $Ext(H_3) = \{1\}$. Thus, $T(B; x, y) = x^4 + x^3 + x^2y$.

We use now *Definition 2.16*. As the edges have numbers as labels, we search for the first ordinary edge, which is **2** in this case. We calculate $B -$ **2** and $B/$**2**.

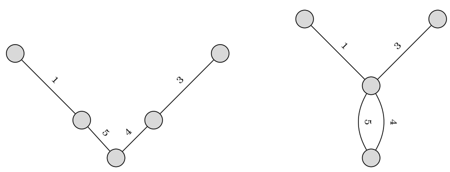

Figure 2.4: Left: $B -$ **2**. Right: $B/$**2**

We notice that $B -$ **2** consists only of bridges, so $T(B -$ **2**$; x, y) = x^4$. For $B/$**2** we repeat the process, the first ordinary edge in this case being **4**.

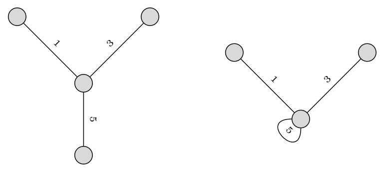

Figure 2.5: Left: $(B - \mathbf{2}) - \mathbf{4}$. Right: $(B/\mathbf{2})/\mathbf{4}$

Our process ends here, as $T((B - \mathbf{2}) - \mathbf{4}; x, y) = x^3$ and $T((B/\mathbf{2})/\mathbf{4}; x, y) = x^2y$. Lastly, $T(B; x, y) = x^4 + x^3 + x^2y$.
For *Definition 2.14*, we compute the power set of the edge set,

$$\mathcal{P}(X) = \{\emptyset, \{1\}, ..., \{1, 2, 3, 4, 5\}\}$$

Next, we calculate the terms of the Tutte Polynomial, which give us the same result as in the other two formulas.

Before we go into more details and theorems, we draw a detailed proof of the equivalency of the deletion/contraction and spanning forest expansion definitions.

Proof [Bol98]: We prove the claim by induction on the number of edges of G.

- *Base step:* We use the empty graph E_n. As it does not have any edges, $t_{00} = 1$ and $t_{ij} = 0$ when $i + j > 0$; thus, $\sum_{i,j} t_{ij}x^iy^i = 1 = T(E_n; x, y)$.
- *Induction step:* Let $G = (V, E)$ be a graph, $E(G) = e_1, ..., e_m; m \geq 1$. The edge set is linearly ordered in the following way: $e_1 > ... > e_m$. We assume the equivalency is true for the graph with $m - 1$ edges. We consider $G' = G - e_m$ and $G'' = G/e_m$, $E(G') = E(G'') = E(G) \setminus e_m$. We denote t'_{ij} to be the number of (i, j)-forests of G', i.e. forests with internal activity i and external activity j, and t''_{ij} - (i, j)-forests of G''. By the claim of the induction step:

$$T(G - e_m; x, y) = \sum_{i,j} t'_{ij}x^iy^i \text{ and } T(G/e_m; x, y) = \sum_{i,j} t''_{ij}x^iy^i$$

  It needs to be proven that: $T(G; x, y) = \sum_{i,j} t_{ij}x^iy^i$
  We differentiate among three cases of the nature of the edge e_m.
  - We assume e_m is a bridge; hence, it is a part of every spanning forest F of G. Specifically, e_m is an internally active edge in every spanning forest. If F is a (i, j)-forest of G, then $F - e_m$ is an $(i - 1, j)$-forest of G'; $t_{ij} = t'_{i-1,j}$. Thus:

$$\sum_{i,j} t_{ij}x^iy^j = \sum_{i,j} t'_{i-1,j}x^iy^j = x\sum_{i,j} t'_{i-1,j}x^{i-1}y^j$$

$$= x\sum_{i,j} t'_{ij}x^iy^j = xT(G - e_m; x, y) = T(G; x, y)$$

- We assume e_m is a loop; hence, it is not a part of any spanning forest F of G, but it is always an externally active edge of it (a loop is a cycle of length 1). If F is a (i,j)-forest of G, then $F - e_m$ is an $(i, j-1)$-forest of G'; $t_{ij} = t'_{i,j-1}$. Thus:

$$\sum_{i,j} t_{ij} x^i y^j = \sum_{i,j} t'_{i,j-1} x^i y^j = y \sum_{i,j} t'_{i,j-1} x^i y^{j-1}$$

$$= y \sum_{i,j} t'_{ij} x^i y^j = yT(G - e_m; x, y) = T(G; x, y)$$

- We assume e_m is an ordinary edge. As e_m is the smallest edge in the edge set of G, it cannot be an externally or internally active edge of any spanning forest F of G. It follows that $t_{ij} = t'_{ij} + t''_{ij}$ and

$$\sum_{i,j} t_{ij} x^i y^j = \sum_{i,j} t'_{ij} x^i y^j + \sum_{i,j} t''_{ij} x^i y^j$$

$$= T(G - e_m; x, y) + T(G/e_m; x, y) = T(G; x, y).$$

This proves the claim for the graph G with m edges, which completes the proof.

□

The proof for the equivalency of the rank/co-rank and deletion/contraction definition is similar to the ones presented for the spanning subgraph and rank generating polynomial, i.e. *Theorem 2.3* and *Theorem 2.11*

Theorem 2.19
Let $G = (V, E)$ be a graph, $T(G; x, y)$ - its Tutte polynomial and $S(G; x, y)$ - its Whitney rank-generating polynomial. The Tutte polynomial is a simple function of the rank-generating polynomial:

$$T(G; x, y) = S(G; x - 1, y - 1)$$

The proof follows directly from the definition of the rank-generating polynomial.

Theorem 2.20
Let $G = (V, E)$ be a graph, $T(G; x, y)$ - its Tutte polynomial and $\mathcal{Z}(G; x, y)$ - its spanning subgraph polynomial. The Tutte polynomial is a simple function of the spanning subgraph polynomial:

$$T(G; x, y) = (x - 1)^{-k(E)} (y - 1)^{-|V|} \mathcal{Z}(G; y - 1, (x - 1)(y - 1))$$

Proof [Tit22]: We prove the theorem using both definitions of the polynomials.

- Firstly, we use the rank/co-rank formula of the Tutte polynomial.

$$T(G; x, y) = (x - 1)^{-k(E)} (y - 1)^{-|V|} \mathcal{Z}(G; y - 1, (x - 1)(y - 1))$$

$$= (x - 1)^{-k(E)} (y - 1)^{-|V|} \sum_{F \subseteq E} (y - 1)^{|F|} [(x - 1)(y - 1)]^{k(F)}$$

$$= \sum_{F \subseteq E} (x-1)^{k(F)-k(E)} (y-1)^{|F|-|V|+k(F)}$$

$$= \sum_{F \subseteq E} (x-1)^{r(E)-r(F)} (y-1)^{n(F)}$$

- Secondly, we use the spanning trees expansion formula.

$$\mathcal{Z}(G; x, y) = x^{|V|-k(E)} y^{k(E)} \sum_{H \in \tau(G)} (1 + \frac{y}{x})^{int(H)} (1+x)^{ext(H)}$$

We substitute $s = 1 + \frac{y}{x}$ and $t = 1 + x$, which provides $x = t - 1$ and $y = (s-1)(t-1)$.

$$\mathcal{Z}(G; t-1, (s-1)(t-1)) =$$

$$= (t-1)^{|V|-k(E)} [(s-1)(t-1)]^{k(E)} \sum_{H \in \tau(G)} s^{int(H)} t^{ext(H)}$$

$$= (t-1)^{|V|-k(E)} [(s-1)(t-1)]^{k(E)} T(G; x, y)$$

The equality yields:

$$T(G; x, y) = (t-1)^{k(E)-|V|} (s-1)^{-k(E)} (t-1)^{-k(E)} \mathcal{Z}(G; t-1, (s-1)(t-1))$$

which proves the theorem.

□

Example 2.21
We use the star graph S_3 again as we have calculated both the spanning subgraph and the rank-generating polynomials for it. We show that they are all equivalent.
Using *Definition 2.16*, as the graph has only 3 bridges, we attain: $T(S_3; x, y) = x^3$. Correspondingly, the graph has only one spanning tree which has 3 internally active edges. We start from the rank-generating polynomial:

$$T(S_3; x, y) = S(S_3; x-1, y-1) = (x+1-1)^3 = x^3.$$

We consider now the spanning subgraph polynomial:

$$T(S_3; x, y) = (x-1)^{-k(S_3)} (y-1)^{-|V(S_3)|} \mathcal{Z}(S_3, y-1, (x-1)(y-1) =$$

$$= (x-1)^{-1}(y-1)^{-4}(x-1)(y-1)((x-1)(y-1)+y-1)^3 = (y-1)^{-3}((y-1)(x-1+1))^3 = x^3.$$

As shown, all definitions and polynomials mentioned before lead to the same Tutte polynomial.

3 Evaluations of Tutte Polynomial

3.1 Spanning Subgraphs

Theorem 3.1
Let $G = (V, E)$ be a graph and $\tau(G)$ be the set of all spanning forests of G, then

1. $T(G; 1, 1) = |\tau(G)|$
2. $T(G; 2, 1)$ *equals to the number of forests (acyclic edge subsets) of* G
3. $T(G; 1, 2)$ *equals to the number of spanning subgraphs of* G *that have the same number of components as the original graph*
4. $T(G, 2, 2) = 2^{|E|}$*, i.e. number of spanning subgraphs.*

Proof [Bol98]: All these observations follow from the rank/co-rank definition of the Tutte polynomial.

1. $T(G; 1, 1) = \sum_{F \subset E} 0^{r(E)-r(F)} 0^{n(F)} = |\{F : F \in E, r(E) = r(F)$ and $n(F) = 0\}|$. $r(E) = r(F)$ means that (V, F) has the same number of components as G, $n(F) = 0$ implies that (V, F) is a forest; thus $T(G; 1, 1)$ is the number of spanning forests of G.
2. $T(G; 2, 1) = \sum_{F \subset E} 1^{r(E)-r(F)} 0^{n(F)} = |\{F : F \in E$ and $n(F) = 0\}|$. As mentioned, $n(F) = 0$ implies that (V, F) is a forest, thus $T(G; 2, 1)$ is the number of forests of G.
3. $T(G; 1, 2) = \sum_{F \subset E} 0^{r(E)-r(F)} 1^{n(F)} = |\{F : F \in E$ and $r(E) = r(F)\}|$. $r(E) = r(F)$ means that (V, F) has the same number of components as G, thus $T(G; 1, 2)$ is the number of spanning subgraphs of G that have the same number of components as G.
4. $T(G; 2, 2) = \sum_{F \subset E} 1^{r(E)-r(F)} 1^{n(F)} = |\{F : F \in E\}| = 2^{|E|}$.

□

3.2 Orientations and Score vectors

Theorem 3.1 can be rewritten using orientations and score vectors of a graph.
An **orientation** $\vec{G} = (V, E)$ of a graph $G = (V, E)$ is an assignment of a direction to every edge in the graph. A **score vector** of an orientation is the vector $(s_1, s_2, ..., s_n)$ given that vertex i has an outdegree s_i in $\vec{G}$.

Theorem 3.2
Let $G = (V, E)$ be a graph and $T(G; x, y)$ its Tutte polynomial. Then:

1. $T(G; 2, 0)$ *is the number of acyclic orientations of the graph, i.e. orientations that have no oriented cycles*
2. $T(G; 0, 2)$ *is the number of totally cyclic orientations of* G*, i.e. orientations in which every edge belongs to at least one cycle.*
3. $T(G; 1, 0)$ *is the number of acyclic orientation that have one predefined source* v*.*
4. $T(G, 2, 1)$ *is the number of score vectors of orientations of the graph.*

Details for the proof of this theorem can be found in Ellis-Monaghan and Merino [EM11]. The evaluations at $T(G;1,-1)$ and $T(G;2,-1)$ and their results can be found in Goodall et al. [Goo+13]. More conjectures that provide inequalities between the evaluations of the Tutte polynomial are revealed and studied in Jackson [Jac10].

3.3 Derivatives of Tutte Polynomial

Theorem 3.3

Let $G = (V, E)$ be a graph with its edges linearly ordered. The following derivative with respect to y of the $T(G;1,y)$ is the number of spanning subgraphs H of G that contain exactly one cycle and have the same number of components as the original graph:

$$\left.\frac{\partial T(G;1,y)}{\partial y}\right|_{y=1}$$

Proof [Tit22]: From the spanning forests definition of the Tutte polynomial, we have:

$$\frac{\partial}{\partial y}T(G;1,y) = \sum_{H\in\tau(G)} ext(H)y^{ext(H)}$$

Thus:

$$\left.\frac{\partial}{\partial y}T(G;1,y)\right|_{y=1} = \sum_{H\in\tau(G)} ext(H)$$

According to *Theorem 2.6*, for every graph J with one cycle, there exists a unique spanning forest H, a unique edge $e \in Int(H)$ with $J = H + e$. Thus, each spanning forest generates precisely $ext(H)$ unicyclic graphs. □

3.4 Closed forms and recurrence relations for the Tutte Polynomial

We mention certain closed forms for some classes of graphs, see [Mat23]:

- For a cycle graph C_n: $T(C_n; x, y) = y + x + x^2 + ... + x^{n-1}$
- For a forest G: $T(G; x, y) = x^{|E(G)|}$
- For a path graph P_n: $T(P_n; x, y) = x^{n-1}$

We specify recurrence relations for the Tutte polynomial of some classes of graphs:

- For a cycle graph C_n: $T(C_n; x, y) = (x-1)T(C_{n-1}; x, y) - xT(C_{n-2}; x, y)$
- For the path graph P_n: $T(P_n; x, y) = xT(P_{n-1}; x, y)$
- For the wheel graph W_n:

$$T(W_n; x, y) = (x + y + 2)T(W_{n-1}; x, y)-$$

$$-(x + 1)(y + 1)T(W_{n-2}; x, y) + xyT(W_{n-3}; x, y)$$

.

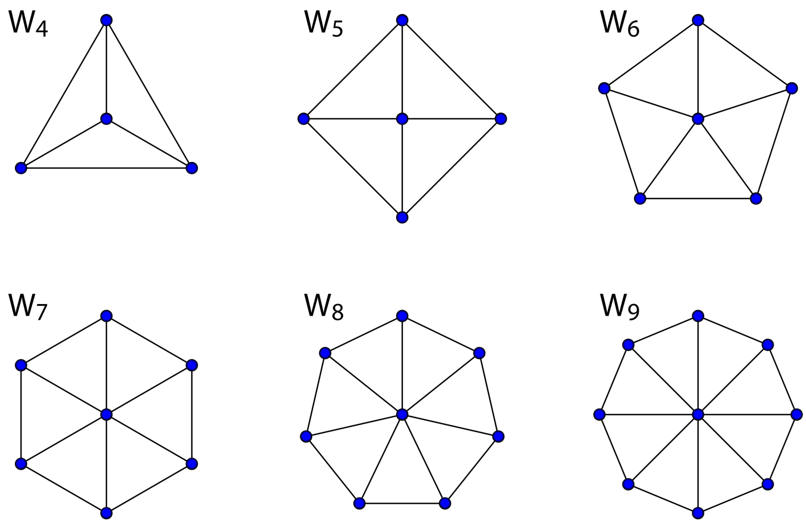

Figure 3.1: Wheel graphs [Exc23]

Figure 3.2: Path graph [Wik23]

4 Properties

4.1 Planar graphs

Let $G = (V, E)$ be a graph. G is called **planar** if we can draw it the plane such that there are no edge crossings.The **dual graph** G^* has the faces of G as its vertices. We connect two vertices of G^* with n edges if the corresponding faces are connected by n boundary edges in G. Let f denote the number of faces of a graph G, then $V(G^*) = f$ and $E(G^*) = E(G)$.
Moreover, if $F \subseteq E$ is a spanning forest in G, then $E \setminus A$ is a spanning forest in G^*. There also exists a bijective map between the spanning forests of a graph and its dual that exchanges the internal and external activity of them, meaning $t_{ij} = t^*_{ij}$, see Biggs [Big74].
Regarding the deletion/contraction formula, we observe that $(G/e)^* \cong G^* - e$ and $(G-e)^* \cong G^*/e$, see Goodall [Goo13]. This gives us the following relation between the Tutte polynomial of a graph and its dual.

Theorem 4.1
Let G be a planar graph and G^* its dual graph. Then:

$$T(G; x, y) = T(G^*; y, x).$$

Example 4.2
We consider the *Cycle graph* $C_3 = G$ and its dual, G^*.

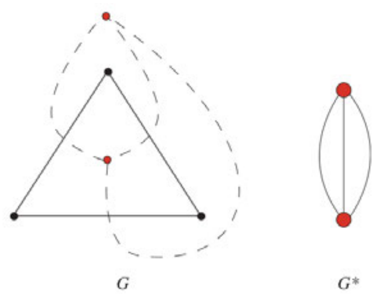

Figure 4.1: Cycle graph and its dual [Wei23]

As seen in *Chapter 3.4*, the Tutte polynomial for the cycle graph is : $T(G; x, y) = y + x + x^2$.
We focus on the dual.

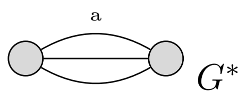

Figure 4.2: Dual of the cycle graph G

We use the recursive formula, starting with the edge a. We delete and contract it and analyze the new graphs.

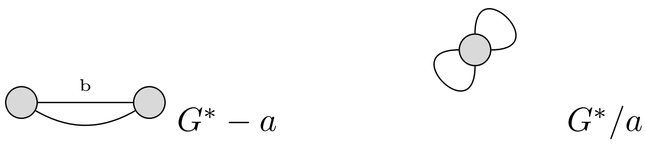

Figure 4.3: Left: $G^* - a$. Right: G^*/a

The polynomial for the G^*/a can be calculated directly, as it consists only of loops: $T(G^*/a; x, y) = y^2$. For the graph $G^* - a$, we use the formula again for the edge b.

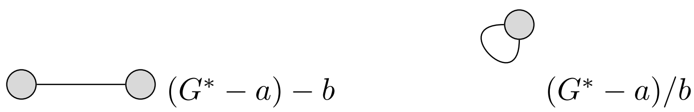

Figure 4.4: Left: $(G^* - a) - b$. Right: $(G^* - a)/b$

The graph $(G^* - a) - b$ consists of only one edge, thus $T((G^* - a) - b; x, y) = x$; and the graph $(G^* - a)/b$ has one loop: $T((G^* - a)/b; x, y) = y$. Adding all polynomials together, we obtain: $T(G; x, y) = y^2 + x + y$. From *Theorem 4.1*, we have:

$$T(G^*; x, y) = T(G; y, x) = x + y + y^2$$

which is indeed the case.

4.2 Tutte polynomial of components

Theorem 4.3
Let G and H be graphs and $T(G; x, y)$, $T(H; x, y)$ - their respective Tutte polynomials. Then:

$$T(G \cup H; x, y) = T(G; x, y)T(H; x, y)$$

and

$$T(G * H; x, y) = T(G; x, y)T(H; x, y)$$

Proof: The formulas follow from the equivalent ones using the spanning subgraph polynomial and its equivalency to the Tutte polynomial.
Still, we sketch the proof for the Tutte polynomial of the one point join. As mentioned previously, in the case of $G * H$, $V(G * H) = V(G) + V(H) - 1$ and $k(G * H) = k(G) + k(H) - 1$. Using *Theorem 2.18* , for a graph $G = (V, E)$ and *Theorem 2.8* it is given that:

$$T(G; x, y) = (x - 1)^{-k(E)}(y - 1)^{-|V|}\mathcal{Z}(G; y - 1, (x - 1)(y - 1))$$

Which yields:

$$T(G*H;x,y) = (x-1)^{-k(G*H)}(y-1)^{-|V(G*H|)}\mathcal{Z}(G*H;y-1,(x-1)(y-1))$$
$$= (x-1)^{-k(G)-k(H)+1}(y-1)^{-|V(G)|-|V(H)|+1}$$
$$\frac{1}{(x-1)(y-1)}\mathcal{Z}(G;y-1,(x-1)(y-1))\mathcal{Z}(H;y-1,(x-1)(y-1))$$
$$= \frac{(x-1)(y-1)}{(x-1)(y-1)}(x-1)^{-k(G)-k(H)}(y-1)^{-|V(G)|-|V(H)|}$$
$$\mathcal{Z}(G;y-1,(x-1)(y-1))\mathcal{Z}(H;y-1,(x-1)(y-1))$$
$$= (x-1)^{-k(G)}(y-1)^{-|V(G)|}\mathcal{Z}(G;y-1,(x-1)(y-1))$$
$$(x-1)^{-k(H)}(y-1)^{-|V(H)|}\mathcal{Z}(H;y-1,(x-1)(y-1))$$
$$= T(G;x,y)T(H;x,y).$$

□

Example 4.4
We show the theorem for the graph $G * H$, where G is the cycle graph C_3 and H consists of a bridge and a loop, see *Figure 4.5*.

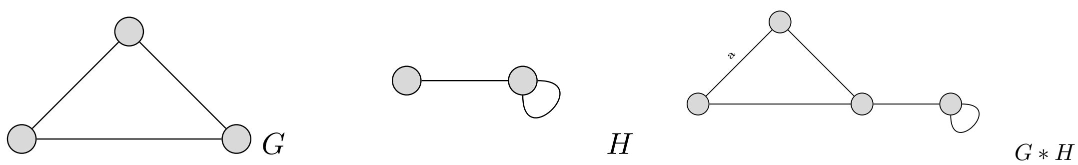

Figure 4.5: G, H and $G * H$.

We calculate the polynomial of $G * H$ using the recursive formula, choosing a as our first ordinary edge. We delete and contract it.

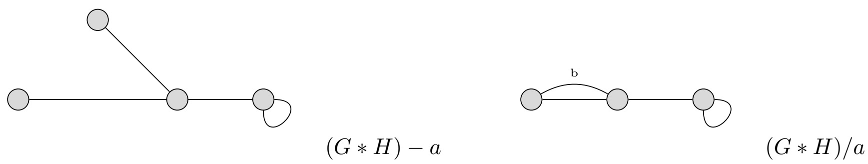

Figure 4.6: Left: $(G * H) - a$. Right: $(G * H)/a$.

The graph $(G * H) - a$ consists only of 3 bridges and one loop, which means $T((G * H) - a; x, y) = x^3y$. For the contracted graph, we choose the edge b and repeat the process.

$((G * H)/a) - b$ $((G * H)/a)/b$

Figure 4.7: Left: $((G * H)/a) - b$. Right: $((G * H)/a)/b$.

Now, both graphs consist only of loops and bridges, thus: $T(((G * H)/a) - b; x, y) = x^2 y$ and $T(((G * H)/a)/b; x, y) = xy^2$. Lastly, the Tutte polynomial for $G * H$ is :

$$T(G * h; x, y) = T((G * H) - a; x, y) + T(((G * H)/a) - b; x, y) + T(((G * H)/a)/b; x, y) =$$

$$= x^3 y + x^2 y + xy^2.$$

Using *Theorem 4.3*, the Tutte polynomial of a one-point join is the product of the respective graphs, in our case:

$$T(G; x, y) = y + x + x^2 \text{ and } T(H; x, y) = xy,$$

which yields:

$$T(G * H; x, y) = T(G; x, y)T(H; x, y) = (y + x + x^2)xy = xy^2 + x^2 y + x^3 y.$$

As we have seen, both polynomials coincide.

4.3 Identities for the coefficients of the Tutte Polynomial

Theorem 4.5
Let $G = (V, E)$ be a graph and $T(G; x, y) = \sum_{i,j} t_{ij} x^i y^j$ its Tutte polynomial. If $|E(G)| > 0$, then $t_{00} = 0$ If $|E(G)| \geq 2$, then $t_{10} = t_{01}$.

Theorem 4.6
Let $G = (V, E)$ be a graph, $E(G) \geq m$. Brylawski [Bry10] proved the following:

$$\sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^j \binom{k-i}{j} t_{ij} = 0$$

where $k = 0, 1, ..., m - 1$.

Theorem 4.7
Let $G = (V, E)$ be a graph without loops, $k(G) = 2$, $E(G) = m$, $V(G) = n$ and $T(G; x, y) = \sum_{i,j} t_{ij} x^i y^j$ its Tutte polynomial. Then:

$$\forall i, 1 \leq i \leq n - 1 : t_{i0} > 0 \textit{ and } \forall j, 1 \leq i \leq m - n + 1 : t_{0j} > 0$$

Theorem 4.8
Let $G = (V, E)$ be a graph without loops, is not a cycle or contains parallel edges, $k(G) = 2$. Then $t_{11} > 0$.

The proofs for these theorems can be found in Bollobás [Bol98] and Goodall [Goo13]. Expressions for further coefficients can be found in Gong, Jin, and Li [GJL20].

5 Universality of Tutte Polynomial

A **graph invariant** is the property of graphs which is preserved by isomorphisms, i.e. a function f such that $f(G_1) = f(G_2)$ whenever $G_1 \cong G_2$. A **graph polynomial** is a graph invariant whose values are polynomials, i.e. the image of the graph invariant lies in a polynomial ring. A graph G is called a **minor** of a graph H, if G can be formed by deletion of edges or vertices, contraction of edges of H. A class of graphs $\mathcal{G}$ is **minor-closed** if every minor of G is in $\mathcal{G}$, $\forall G \in \mathcal{G}$, see [Eri09].

Definition 5.1
Let $\mathcal{G}$ be a minor-closed class of graphs. A graph invariant f from $\mathcal{G}$ to a commutative ring $\mathcal{R}$ with unity is a **generalized Tutte-Gröthendieck invariant** or **T-G invariant** if:

- $f(E_1)$ is the unity of $\mathcal{R}$
- $\exists a, b \in \mathcal{R} : \forall G \in \mathcal{G}, \forall e \in G : f(G) = af(G - e) + bf(G/e)$;
- $\forall G, H \in \mathcal{G}$, whenever $G \cup H \in \mathcal{G}$ or $G * H \in \mathcal{G}$: $f(G \cup H) = f(G)f(H)$ and $f(G * H) = f(G)f(H)$

The Tutte polynomial is a T-G invariant, in fact, it is the *only* T-G invariant, i.e. all others are an evaluation of it, see Ellis-Monaghan and Merino [EM11].

Theorem 5.2
Let $\mathcal{G}$ be a minor-closed class of graphs, $\mathcal{R}$ a commutative ring with unity and $a, b, x_0, y_0 \in \mathcal{R}$. Then there is a unique T-G invariant $f : \mathcal{G} \to \mathcal{R}$ such that:

$$f(G) = a^{|E(G)|-r(E(G))} b^{r(E(G))} T(G; \frac{x_0}{b}, \frac{y_0}{a})$$

where $f(B) = x_0$ and $f(L) = y_0$.

The proof of the theorem is by induction on the number of ordinary edges using the deletion/contraction definition of the Tutte polynomial and can be found in Bollobás [Bol98].

6 The Tutte Polynomial of Rooted Graphs

A **rooted graph** is a a graph in which a specific vertex is taken as the *root*. We denote it as G_v, where v is the root. Given that the graph is a tree, a **rooted subtree** is a subgraph that is also a tree and has the same root as the original graph.

Definition 6.1
Let $G_v = (V, E)$ be a rooted graph. Then, for every subset A of the edge set, $r(A)$ is defined as follows:

$$r(A) = \max_{F \subseteq A} \{|F| : F - \text{rooted subtree of } T\}$$

Definition 6.2
Let $G_v = (V, E)$ be a rooted graph. Then the Tutte polynomial for it is defined in the following way:

$$T(G_v; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}$$

But in this case, $r(A)$ is defined as in the previous theorem and it is called the *branching rank*.

An interesting concept is presented in Gordon [Gor] where the deletion/ contraction formula for the Tutte polynomial of a rooted graph is presented.

Theorem 6.3
Let $G_v = (V, E)$ *be a rooted graph,* $E(G_v) = n$, e *be an adjacent edge to* v*. Then:*

$$T(G_v; x, y) = T(G_v/e; x, y) + (x-1)^{r(G_v)-r(G_v-e)} T(G_v - e; x, y).$$

Gordon states that the term $(x-1)^{r(G_v)-r(G_v-e)}$ is not present in the definition for ordinary graphs, as $r(G) = r(G-e)$, unless e is a bridge. The detailed proof, as well as more theorems and propositions about the Tutte polynomial of rooted graphs and trees can be found in Gordon [Gor] and Chaudhary and Gordon [CG91].

7 Polynomials that can be derived from the Tutte Polynomial

We introduce other polynomials that can be derived from the Tutte polynomial substituting x and y accordingly, with some possible prefactor. We call them **specializations of the Tutte Polynomial**.

7.1 Chromatic Polynomial

We start with the best known and studied polynomial in graph theory: *the chromatic polynomial*.
Let $G = (V, E)$ be a graph and λ a positive integer. A map $\phi : V(G) \to \{1, ..., \lambda\}$ is called a **vertex coloring** of G , such that it allocates a color to every vertex of the graph. ϕ is called a **proper(admissible) coloring** of G if $\forall i, j \in E(G) : \phi(i) \neq \phi(j)$.
The **chromatic number** $\mathcal{X}(G)$ is the smallest number of colors used for a proper coloring of G.

Definition 7.1
Let $G = (V, E)$ be a graph. The **chromatic polynomial** of G is

$$\mathcal{X}(G; \lambda) = \sum_{F \subseteq E} (-1)^{|F|} \lambda^{k(F)}.$$

Theorem 7.2
Let G be a graph with its chromatic polynomial $\mathcal{X}(G; \lambda)$. Then:

- $\mathcal{X}(E_n; \lambda) = \lambda^n$
- $\mathcal{X}(G; \lambda) = 0, \forall \lambda$*, if G has a loop*
- $\mathcal{X}(K_n; \lambda) = \lambda(\lambda - 1)...(\lambda - n + 1)$
- $\mathcal{X}(G; \lambda) = \frac{x-1}{x} \mathcal{X}(G - e; \lambda)$*, if e is a bridge*
- *if we let e be any edge of G, then:*

$$\mathcal{X}(G; \lambda) = \mathcal{X}(G - e; \lambda) - \mathcal{X}(G/e; \lambda).$$

The detailed proof of this theorem can be found in Tittmann [Tit22].

Theorem 7.3
Let G and H be graphs and $\mathcal{X}(G; \lambda), \mathcal{X}(H; \lambda)$ their respective chromatic polynomials. Then:

$$\mathcal{X}(G \cup H; \lambda) = \mathcal{X}(G; \lambda)\mathcal{X}(H; \lambda)$$

and

$$\mathcal{X}(G * H; \lambda) = \frac{1}{\lambda} \mathcal{X}(G; \lambda)\mathcal{X}(H; \lambda)$$

The proof of this theorem can be found in Github [Git23].
We can see that the chromatic polynomial satisfies the universality property, thus it can written as a specialisation of the Tutte polynomial.

Theorem 7.4
Let $G = (V, E)$ be a graph, $\mathcal{X}(G;\lambda)$ its chromatic polynomial and $T(G;x,y)$ its Tutte polynomial. Then:

$$\mathcal{X}(G;\lambda) = (-1)^{r(E)}\lambda^{k(E)}T(G;1-\lambda,0).$$

Proof [EM11]: We start with the Tutte polynomial:

$$\begin{aligned}
T(G;1-\lambda,0) &= \sum_{F\subseteq E}(1-\lambda-1)^{k(F)-k(E)}(0-1)^{k(F)+|F|-|V|}\\
&= \sum_{F\subseteq E}(\frac{\lambda}{-1})^{k(F)-k(E)}(-1)^{k(F)+|F|-|V|}\\
&= \sum_{F\subseteq E}(-1)^{k(E)-k(F)+k(F)+|F|-|V|}\lambda^{k(F)-k(E)}\\
&= (-1)^{k(E)-|V|}\lambda^{-k(E)}\sum_{F\subseteq E}(-1)^{|F|}\lambda^{k(F)}\\
&= (-1)^{-r(E)}\lambda^{-k(E)}\mathcal{X}(G;\lambda).
\end{aligned}$$

□

The Beta Invariant

An important invariant of the chromatic polynomial worth mentioning is the **beta invariant**.

Definition 7.5
Let $G = (V, E)$ be a graph with $E(G) \geq 2$. The **beta invariant** or **chromatic invariant** is defined in the following way:

$$\beta(G) = (-1)^{r(E)}\sum_{F\subseteq E}(-1)^{|F|}r(F).$$

The invariant is related to the trees expansion formula of the Tutte polynomial.

Theorem 7.6
Let $G = (V, E)$ be a graph with $E(G) \geq 2$ and $T(G;x,y) = \sum_{i,j} t_{ij}x^iy^j$ its Tutte polynomial. Then $t_{01} = t_{10} = \beta(G)$.

Theorem 7.7
Let $G = (V, E)$ be a graph and $\mathcal{X}(G; \lambda)$ its chromatic polynomial. Then the derivative of the polynomial with $\lambda = 1$ is:

$$\mathcal{X}'(G; 1) = (-1)^{r(E)+1}\beta(G).$$

The proof for the previous two theorems can be found in Bollobás [Bol98]

7.2 Bad Colouring Polynomial

A generalization of the chromatic polynomial is the counting of not only the proper colorings, but of *all* possible colorings of a graph G. In this situation, we have edges that connect vertices with the same colour. We name them *bad edges*, this is where the polynomial gets its name of *bad coloring polynomial*.

Definition 7.8
Let $G = (V, E)$ be a graph. The **bad colouring polynomial** of G is:

$$\mathcal{B}(G; \lambda, t) = \sum_j b_j(G; \lambda)t^j$$

where $b_j(G; \lambda)$ is the number of λ-colourings with j bad edges.

Remark 7.9
The bad colouring polynomial can be written in the following way:

$$\mathcal{B}(G; \lambda, t+1) = \sum_{\phi: V \to [\lambda]} (t+1)^{|b(\phi)|}.$$

where ϕ is the map that describes the colouring of the vertices and $b(\phi)$ is the set of bad edges of it.

Theorem 7.10
Let $G = (V, E)$ be a graph, $T(G; x, y)$ its Tutte polynomial and $\mathcal{B}(G; \lambda, t+1)$ its bad coloring polynomial. Then:

$$\mathcal{B}(G; \lambda, t+1) = t^{r(E)}\lambda^{k(E)}T(G; \frac{\lambda + t}{t}, t+1)$$

Proof [EM11]: We manipulate further the bad coloring polynomial using the chromatic polynomial.

$$\mathcal{B}(G; \lambda, t+1) = \sum_{\phi: V \to [\lambda]} (t+1)^{|b(\phi)|}$$

$$= \sum_{\phi: V \to [\lambda]} \sum_{F \subseteq b(\phi)} t^{|F|} = \sum_{F \subseteq E} \sum_{\phi: V \to [\lambda], F \subseteq b(\phi)} t^{|F|} = \sum_{F \subseteq E} t^{|F|}\lambda^{k(F)}$$

We use the Tutte polynomial rank/co-rank formula and specialize it for the bad coloring polynomial:

$$\mathcal{B}(G;\lambda,t+1)=\sum_{F\subseteq E}t^{|F|}\lambda^{k(F)}$$
$$=t^{r(E)}\lambda^{k(E)}\sum_{F\subseteq E}t^{|F|-r(F)}(\frac{\lambda}{t})^{r(E)-r(F)}$$
$$=t^{r(E)}\lambda^{k(E)}T(G;1+\frac{\lambda}{t},t+1)$$

□

7.3 Flow Polynomial

Let $\vec{G}=(V,E)$ be a graph with a fixed orientation and N a finite abelian group with the identity element 0 and order k. A **flow** on $\vec{G}$ is the map $f:E(\vec{G})\to N$ such that the *flow conversation law* or *Kirchhoff's law* is satisfied for any vertex v in $V(G)$:

$$\sum_{e\in E^{-}(v)}f(e)=\sum_{e\in E^{+}(v)}f(e)$$

where $E^{+}(v)$ is the set of all oriented edges starting in v and $E^{-}(v)$ - the set of all oriented edges ending in v, see Sekine and Zhang [SZ97], Ren and Qian [RQ19] and Chen and Yang [CY09].
A **nowhere zero flow** is the map N that never takes the value 0, i.e. $\forall e\in E(\vec{G}):f(e)\neq 0$. We denote $\mathcal{X}^*(G;k)$ as the **flow polynomial** of a graph, i.e. the number of nowhere zero k-flows.

Definition 7.11
Let $G=(V,E)$ be a graph. The **flow polynomial** of G is

$$\mathcal{X}^*(G;k)=\sum_{F\subseteq E}(-1)^{|E(G)|-|F|}k^{|F|-|V|+k(F)}.$$

Theorem 7.12
Let $G=(V,E)$ be a graph, $e\in E(G)$, $\mathcal{X}^*(G;k)$ its flow polynomial. Then:

$$\mathcal{X}^*(G;k)=\begin{cases}0 & \textit{if } e \textit{ is a bridge,}\\ (k-1)\mathcal{X}^*(G-e;k) & \textit{if } e \textit{ is a loop,}\\ -\mathcal{X}^*(G-e;k)+\mathcal{X}^*(G/e;k) & \textit{if } e \textit{ is an ordinary edge.}\end{cases}$$

The proof for this theorem can be found in Bollobás [Bol98].
Combining the last theorem with the universality property of the Tutte polynomial, yields the following theorem proposed by W. Tutte.

Theorem 7.13
Let $G = (V, E)$ be a graph, $\mathcal{X}^*(G; k)$ its flow polynomial and $T(G; x, y)$ its Tutte polynomial. The flow polynomial is a specialisation of the Tutte polynomial in the following way:

$$\mathcal{X}^*(G; k) = (-1)^{n(E)} T(G; 0, 1 - k).$$

Proof: We start with the Tutte polynomial:

$$T(G; 0, 1 - k) = \sum_{F \subseteq E} (-1)^{k(F) - k(E)} (1 - k - 1)^{k(F) + |F| - |V|}$$

$$= \sum_{F \subseteq E} (-1)^{k(F) - k(E)} (\frac{k}{-1})^{k(F) + |F| - |V|}$$

$$= \sum_{F \subseteq E} (-1)^{k(F) - k(E) - k(F) - |F| + |V|} k^{k(F) + |F| - |V|}$$

$$= (-1)^{-|E(G)| + |V| - k(E)} \sum_{F \subseteq E} (-1)^{|E(G)| - |F|} k^{k(F) + |F| - |V|}$$

$$= (-1)^{-n(E)} \mathcal{X}^*(G; k).$$

□

7.4 Reliability Polynomial

Another polynomial that is worth mentioning is the probability of encountering a connected subgraph of a graph G.

Definition 7.14
Let $G = (V, E)$ be a connected graph, $E(G) = m$, $V(G) = n$. We consider each edge to be active with a chosen probability p. The **(all terminal) reliability polynomial** of G is:

$$R(G; p) = \sum_{F \subseteq E} [(V, F) connected] p^{|F|} (1 - p)^{|E| - |F|}$$

$$= \sum_{k=0}^{m-n+1} g_k p^{k+n-1} (1 - p)^{m-k-n+1}$$

where g_k denotes the number of spanning connected subgraphs of G with $|F| = k + n - 1$

Theorem 7.15
Let $G = (V, E)$ be a connected graph, $E(G) = m$, $V(G) = n$. Then:

$$R(G; p) = p^{n-1} (1 - p)^{m-n+1} T(G; 1, \frac{1}{1 - p}).$$

Proof[EM11]: We rewrite the rank/co-rank definition of the Tutte polynomial using g_k and substituing $x = 1$ and $y = y + 1$. As the part of the Tutte polynomial using x turns to 0, the only terms which will remain will be the ones that have 0^0, i.e. $r(E) = r(F)$ - the spanning connected subgraphs, which yields:

$$T(G; 1, y + 1) = \sum_{k=0}^{m-n+1} g_k y^k$$

Manipulating the reliability polynomial, we obtain:

$$R(G; p) = \sum_{k=0}^{m-n+1} g_k p^{k+n-1}(1 - p)^{m-k-n+1}$$

$$= p^{n-1}(1 - p)^{1-n+m} \sum_{k=0}^{m-n+1} g_k p^k (1 - p)^{-k}$$

$$= p^{n-1}(1 - p)^{1-n+m} \sum_{k=0}^{m-n+1} g_k (\frac{p}{1 - p})^k$$

$$= p^{n-1}(1 - p)^{1-n+m} T(G; 1, \frac{p}{1 - p} + 1)$$

$$= p^{n-1}(1 - p)^{1-n+m} T(G; 1, \frac{1}{1 - p}).$$

□

8 Further research

'One of the most important numerical quantities that can be computed from a graph is the Tutte polynomial', according to Gessel and E.Sagan [GE95]
The Tutte polynomial continues to be a topic for research until today, we mention some further articles and papers that offers more details and notions on the topic.
Firstly, the polynomial can be expanded to matroids. This expansion is vastly studied, see Kook, Reiner, and Stanton [KRS97], Reiner [Rei99], Brylawski [Bry10], Welsh [Wel99] .
A version of the Tutte Polynomial can be considered to be the graded Euler characteristic with which a bigraded chain complex can be build, see Jasso-Hernandez and Rong [JR06].
In Bernardi [Ber08], a new conviction of activity, called the *embedding-activity*, which makes use of the *combinatorial embedding*, is used to present a new equivalent definition of the Tutte polynomial as a generating function of spanning trees. More research on embedded graphs, especially the ribbon graph, can be found in “Irreducibility of the Tutte polynomial of an embedded graph” [22].
The uniqueness of the Tutte polynomial has also been explored, see Mier and Noy [MN04]. Namely, known classes of graphs, like wheel graphs, ladder graphs, etc.; are shown to be T-unique, that is they are isomorphic due to the fact that they have the same Tutte polynomial.
A multivariante arithmetic Tutte polynomial is introduced in Bränden and Moci [BM14].
The Tutte polynomial has a fascinating application in statistical physics, specifically in the *Potts model* and *Ising model*, see Welsh and Merino [WM00] and M.Porter [MPo15]. Its connection to knot theory can be found in Bollobás [Bol98], Kang et al. [Kan+16], Kaufmann [Kau89] and Goodall [Goo14]. Research on configurations, Merino's theorem and Gessel's formula can be found in Brandt [Bra15] and Biggs [Big99].
An extension to hypergraphs and polymatroids and the definition of a *universal Tutte polynomial* of degree n can be discovered in Bernardi, Kalman, and Postnikov [BKP20].

Python Computation

As the Tutte polynomial contains information about a graph and its properties, its computation has been attempted. Even though the problem of finding an exact calculation is part of the complexity class **#P**, I was able to write the code for the estimation of the polynomial using the recursive and rank/co-rank formulas during my internship. I used a **MultiGraph** class to define a graph with loops or parallel edges.
Firstly, I defined methods as *count_components* and *dfs* (depth first search) for the rank/co-rank definition and return the polynomial as a *sympy* expression. For the recursive formula, I needed methods as *remove_edges, contract_edges, edge_type* and *first_ordinary_edge* to calcultate the *sympy* expression of the polynomial.
To check whether the polynomials were calculated correctly and coincided, I tested 17 graphs. To be specific, I confirmed that the polynomials given by the two formulas were the same and whether the properties, mentioned in *Theorem 3.1*, which I calculated separately, corresponded with the evaluations of the Tutte polynomial. For more details and the detailed code, check Stan [Sta23].

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Statutory Declaration in Lieu of an Oath

I – Natalia Stan – do hereby declare in lieu of an oath that I have composed the presented work independently on my own and without any other resources than the ones given.

All thoughts taken directly or indirectly from external sources are correctly acknowledged.

This work has neither been previously submitted to another authority nor has it been published yet.

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