Frank Simon

Enumerative Combinatorics in the Partition Lattice

MASTERARBEIT

Hochschule Mittweida (FH)

University of Applied Sciences

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Abstract:

The focus of this thesis lies on the application of enumerative combinatorics to the partition lattice Π_n. Some ideas beyond enumerative combinatorics are used to examine the lattice properties of Π_n. The combinatorial aspects involve the counting of chains in the partition lattice with the associated exponential generating functions and a new derivation of the Möbius function in Π_n. Finally different representations of some simple sums ranging over intervals in the partition lattice are discussed.

Contents

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1. Introduction

The thesis on hand represents a combinatorial survey of set partitions and the associated partition lattice. The main aim of this thesis is to compile combinatorial results about set partitions and the partition lattice, that are scattered among the literature available.

The first part briefly discusses partially ordered sets and some elements of lattice theory. Thereafter the notion of the incidence algebra of a poset is introduced. The incidence algebra lies the framework for the counting of chains in the partition lattice and is utilized in order to deduce an alternative proof for the Möbius function in the partition lattice presented in Chapter 4. Finally an important application of the Möbius function in combinatorics the Möbius inversion is introduced, that will reemerge in Chapter 4 under the restriction of multiplicative functions.

Chapter 3 represents the cornerstone of this thesis and introduces the reader to the tools in combinatorics that are necessary to count set partitions and structures in the partition lattice.

Firstly, the notion of a set partition is stated and some combinatorial problems dealing with set partitions are presented. These problems lead to the Bell numbers and the Stirling numbers of the second and the first kind. All numbers are discussed in the necessary depth giving recurrence relations and other properties of them. The Stirling numbers of the second and the first kind will prove to be a recurrent theme of the thesis on hand.

Secondly, the connection points between graph theory and set partitions are identified. The notion of the independent set polynomial $M(G, \lambda)$ and the chromatic polynomial $\chi(G, \lambda)$ are stated and the connection of both are subsumed in a theorem involving the Stirling numbers of the first and the second kind. Both polynomials are connected by the notion of the set of independent set partitions Π_G of a given simple, undirected graph G.

At next the reader becomes acquainted with the Inclusion-Exclusion principle, that is exemplified to the counting problem of surjective functions, which neighbors the Stirling numbers of the second kind. Using the same principle the theorem of Whitney is stated and proven, which leads to an interesting observations on the counting of graphs. Finally an alternative proof of the connection between chains with and without repetitions in a poset is given by the same principle.

In preparation for Section 3.5 the notion of formal power series is briefly discussed and an application of formal power series as generating functions is exemplified in order to deduce a well known formula for the Stirling numbers of the second kind.

The Section 3.5 represents a recurrent theme when considering counting problems in Chapter 4. The notion of labelled classes taken from [13] gives a natural description of counting problems dealing with set partitions and the partition lattice. The Set operator and the associated bivariate exponential generating function $\hat{H}(z,u)$ are applied to counting problems in graph theory that are used in Chapter 4. Some interesting recurrence relations, that count certain classes of graphs, are deduced and discussed. The author presents tables for the numbers under consideration for some small graphs.

A short Section 3.6 familiarizes the reader with the notion of the Bell polynomials. The Bell polynomials are naturally generated by the Set operator defined in Section 3.5.

The Chapter 4 introduces the partition lattice Π_n and discusses the operations of the infimum and supremum presenting an interesting proof for the semimodular inequality in the partition lattice by utilizing a special bipartite graph. As a centerpiece of this thesis the counting of chains in the partition lattice is presented, depending on the results stated in Section 3.5 and Chapter 2. Recurrence relations for the number of chains of a given length with and without repetitions are stated. It is observed that the chains in Π_n of length k with repetitions in Π_n are given by a special k-fold mapping. Using the results in Section 3.7 taken from [8] enables the author to deduce Theorem 58, which implies an interesting and seemingly new derivation for the Möbius function in the partition lattice. Moreover the interesting connection between the Möbius inversion, the Bell polynomials, and the logarithmic Bell polynomials are briefly stated.

The last section of Chapter 4 examines some simple sums in the partition lattice and presents different representations of these sums by the usage of either basic combinatorial arguments or by utilizing exponential generating functions. A brief outlook to generalize the results to arbitrary intervals in Π_n is then given.

Finally the results found in the thesis are summarized and a brief outlook to some unsolved problems is given.

2. Partially ordered sets

The focus of this thesis lies on the partition lattice Π_n. A lattice is a special case of a partially ordered set. Therefore a few remarks about partially ordered sets and lattice theory are summarized in this chapter. The material compiled here is taken from [16],[32],[17], [2], [29], and [5].

2.1. Partially ordered sets

Definition 1. A *partially ordered set* or short *poset* is a set P endowed with a transitive, reflexive and antisymmetric binary relation $\leq$ on P. It is always assumed in this thesis that P is a finite set, if not explicitly stated differently. Readers that are not familiar with relations are referred to [17, Chapter 5].

The relation $x \geq y$ is equivalent to $y \leq x$ and $x < y$ is equivalent to $x \leq y$, but $x \neq y$. Two elements $x, y \in P$ are *comparable* if $x \leq y$ or $y \leq x$ holds, otherwise they are *incomparable*.

Given a poset $(P, \leq_P)$ the subset $Q \subseteq P$ defines a new poset on $(Q, \leq_Q)$ by $x \leq_Q y$ with $x, y \in Q$ iff $x \leq_P y$ in P. This poset is called the by P *induced subposet* on Q.

Certain subposets $Q \subseteq P$ of a given poset P are of special interest, namely intervals, filters, and ideals. Let A be a subset of the poset P then

$$\mathfrak{F}(A) = \{x \in P : x \geq a \text{ for all } a \in A\} \tag{2.1}$$

is the *filter* generated by A. A filter generated by only one element $a \in P$ is a *principal filter* and is denoted by $\mathfrak{F}(a)$. The dual concept of a filter is an *ideal*, which is denoted by $\mathfrak{I}(A)$ and $\mathfrak{I}(a)$, respectively.

Let $x, y \in P$ then the *interval* $[x, y]$ between x and y is defined by

$$[x, y] = \{z \in P : x \leq z \leq y\}. \tag{2.2}$$

The interval is empty whenever x and y are incomparable.

If $x, y \in P$ then y *covers* x or $x \lessdot y$ if $x < y$ and if there is no $z \in P$, so that $x < z < y$ holds.

The covering relation is useful for depicting an arbitrary poset P in the *Hasse Diagramm*. The elements of $x \in P$ are represented as points $P_x(x_1, x_2)$ in the real plane. Two points P_x, P_y with $x, y \in P$ are connected by a line, iff $x \lessdot y$ such that the coordinates of the points $P_x(x_1, x_2)$ and $P_y(y_1, y_2)$ satisfy $x_2 \leq y_2$. The Hasse Diagramm contains all information about the poset. An example for a Hasse Diagramm is given in Figure 2.1.

A poset P has a *minimum* if there is an element $\hat{0} \in P$, so that $\hat{0} \leq x$ holds for all $x \in P$ and analogous a *maximum* if there is an element $\hat{1} \in P$, so that $x \leq \hat{1}$ holds for all $x \in P$. The poset depicted in Figure 2.1 has the minimum A, but no maximum.

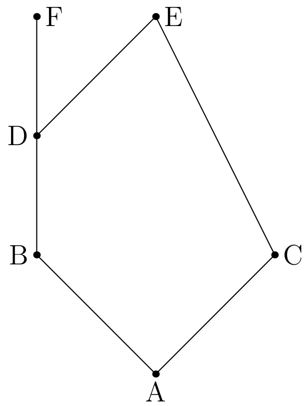

Figure 2.1.: Hasse Diagramm of a partially ordered set

If P has a minimum element $\hat{0}$, then the *atoms* of P are the elements covering $\hat{0}$ and if P has a maximum element $\hat{1}$, then the *coatoms* of P are the elements covered by $\hat{1}$.

An element $x \in P$ is a *minimal* element if there is no $y \in P$ with $y < x$. Similarly an element $x \in P$ is a *maximal* element if there is no $y \in P$ with $x < y$. The example poset in Figure 2.1 has the minimal element A and the two maximal elements F and E.

A subset C of a poset P is called a *chain* if all elements of C are comparable in P. Let $x, y \in P$ with $x \le y$ and C a chain and $x \le z \le y$ for all $z \in C$, then C is called a x-y-chain. A x-y-chain is *maximal* if there is no $z' \in P \setminus C$, so that $C \cup \{z'\}$ is still a x-y-chain. The *length* of a chain C is defined by $|C| - 1$.

A sequence $\{x_k\}_{k=0}^{n}$ of $n+1$ elements of a poset P is called a *chain with repetitions* of length n if $x_i \le x_{i+1}$ for all i with $0 \le i \le n-1$ holds and is denoted by $x_0 \le x_1 \le \ldots \le x_n$. If the sequence satisfies the strict inequality $x_0 < x_1 < \ldots < x_n$, then it is called a *chain without repetitions* of length n.

Assume that P is a poset with minimum $\hat{0}$. If it is possible to define a function $r : P \to \mathbb{N} \cup \{0\}$, so that

$$r(\hat{0}) = 0 \tag{2.3}$$

$$r(y) = r(x) + 1 \text{ for all } x, y \in P \text{ with } x \lessdot y \tag{2.4}$$

is satisfied, then r is called a *rank function* on P. Note that the existence of a rank function in a poset with a minimum element is equivalent to the condition that all maximal x-y-chains in P have the same length.

Let P a poset and $x, y \in P$ then an element $z \in P$ is an *upper bound* for x and y if $x, y \le z$ is satisfied. An upper bound $z \in P$ of $x, y \in P$ is a *least upper bound*[1] of x and y if every upper bound w of x and y satisfies $z \le w$. If a *least upper bound* of two elements $x, y \in P$ exists it is unique and is denoted by $x \vee y$. In a similar fashion the notion of a *greatest lower bound*[2] is introduced which is denoted by $x \wedge y$.

[1] Also called supremum or join
[2] Also called infimum or meet

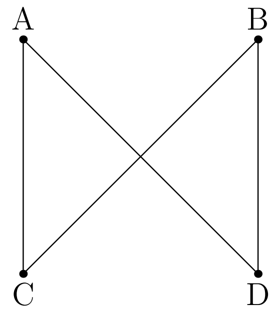

Figure 2.2.: Partially ordered set that is not a lattice

Many posets can be constructed from smaller posets as building blocks. The notion of the product order allows such a construction, see also Theorem 51.

Definition 2. Let P, Q be posets. The *product order* $P \times Q$ is the poset defined on the ordered pairs $(p, q), (r, s) \in P \times Q$ so that $(p, q) \leq (r, s)$ iff $p \leq_P r$ and $q \leq_Q s$.

Sometimes it is possible to find an order preserving and bijective mapping between two posets. In this case one can conclude order theoretic properties from one of the posets that are also valid in the other one.

Definition 3. Let P and Q be posets then P and Q are *isomorphic*, if there is a bijective and order preserving mapping $\phi : P \to Q$ with $x \leq_P y$ iff $\phi(x) \leq_Q \phi(y)$. It is written $P \simeq Q$ if two posets P and Q are isomorphic.

2.2. Lattices

In Chapter 4 it is shown, that Π_n is a lattice. Therefore some theorems and definitions of lattice theory are compiled here. All the theorems listed in this section can be retrieved from [2], [29], and [5].

Definition 4. A *lattice* L is a poset in which every two elements $x, y \in L$ have a least upper bound and a greatest lower bound.

For arbitrary posets this condition does not always hold, as can be seen from the Hasse Diagramm of the poset depicted in Figure 2.2. The least upper bound of C and D does not exist, as A and B are the only upper bounds of C and D, but neither $A \leq B$ nor $B \leq A$ holds.

Theorem 5. Let L be a lattice and $x, y, z \in L$ then the following four assertions hold.

1. $\wedge$ and $\vee$ are commutative, associative, and idempotent operations
2. $x \wedge y = x \Leftrightarrow x \vee y = y \Leftrightarrow x \leq y$
3. $x = x \vee (x \wedge y) = x \wedge (x \vee y)$

4. $x \leq y$ then $x \vee z \leq y \vee z$ and $x \wedge z \leq y \wedge z$

Proof. Only the last two statements are not immeaditely obvious from the definition. The proof of assertion number three is given first.

Due to assertion two for all $x, z \in L$ it is $x = x \wedge z$ iff $x \leq z$. Set $z = x \vee y$ then by definition $x \leq z = x \vee y$ holds and by substituting z by $x \vee y$ in the first equivalence the assertion $x = x \wedge (x \vee y)$ follows. The assertion $x = x \vee (x \wedge y)$ is proven similiar.

By definition $x \vee z$ is the smallest upper bound of x and z, so that for all upper bounds $w \geq x, z$ of x and z it is $w \geq x \vee z$. $y \vee z$ is an upper bound of y and z and by using the assumption $x \leq y$ also an upper bound of x and z. Setting $w = y \vee z$ the assertion $y \vee z \geq x \vee z$ follows. The second part of assertion four is proven likewise. □

Definition 6. A *meet-semilattice* L is a poset, so that $x \wedge y \in L$ exists for all $x, y \in L$.

In Chapter 4 it is shown, that the existence of the infimum in Π_n is easier to prove than the existence of the supremum. In the finite case the theorem from [29, page 103] is therefore practical.

Theorem 7. Let L be a meet-semilattice with $\hat{1}$, then L is lattice.

Proof. Let $x, y \in L$ be elements of the finite meet-semilattice with $\hat{1}$ and define the set $S = \{z \in L : x \leq z, y \leq z\}$ as the set of all upper bounds of x and y, then S must be finite and nonempty as $\hat{1} \in S$. By induction it is concluded that the infimum $\inf S$ of the finitely many elements of S exists, as L is a lattice. Observe that for all $z_1, z_2 \in S$ also $z_1 \wedge z_2 \in S$, as x and y are lower bounds for z_1 and z_2, as $x, y \leq z_1, z_2$, but $z_1 \wedge z_2$ is the greatest lower bound for z_1 and z_2 and therefore $x, y \leq z_1 \wedge z_2 \in S$. Hence

$$x \vee y = \inf S,$$

as $\inf S \leq z$ for all $z \in S$, so that $\inf S$ is the smallest upper bound of x and y. □

A lattice L has furthermore the property that every filter must be a principal filter.

Theorem 8. Let L be a lattice and $\emptyset \neq A \subseteq L$, then there exists an $a \in L$, such that

$$\mathfrak{F}(a) = \mathfrak{F}(A)\,. \tag{2.5}$$

Proof. Recall that $\mathfrak{F}(A) = \{z : z \geq y \text{ for all } y \in A\}$. As L is a lattice and therefore finite there exists an $a \in L$, so that $a = \sup A$. As a is the smallest upper bound for A it follows that $\mathfrak{F}(a) = \mathfrak{F}(A)$. □

Some lattices have the property that all their elements could be constructed out of smaller parts by usage of supremum operation. The partition lattice Π_n introduced in Chapter 4 also shares this property.

Definition 9. A lattice L with a minimum is a *point lattice* if every element $x \in L$ is representable as a supremum of atoms also called the points among the literature concerning lattice theory.

Furthermore it is shown in Chapter 4 that the partition lattice Π_n fulfills the property of semimodularity.

Definition 10. A lattice L is *semimodular* if for all $x, y \in L$

$$x \wedge y \lessdot x \Rightarrow y \lessdot x \vee y \tag{2.6}$$

holds.

The following theorem taken from [2, page 47f.] can be utilized to prove that a lattice is semimodular.

Theorem 11. Let L be a lattice with $\hat{0}$ then L is semimodular iff L has a rank function r and obeys for all $x, y \in L$ the semimodular inequality

$$r(x) + r(y) \geq r(x \vee y) + r(x \wedge y). \tag{2.7}$$

Proof. Sufficiency: Recall that the existence of a rank function r in a poset with $\hat{0}$ is equivalent to the condition that every maximal chain between two elements $x, y \in P$ has the same length.

For $x \lessdot y$ there is only one maximal chain between x and y. Assume that the claim is true for all x and y having a maximal chain of length $t \leq n-1$. So if one maximal chain between x and y has length t, then every maximal chain between x and y has length t.

Assume that $x = c_0 \lessdot c_1 \lessdot \ldots \lessdot c_n = y$ and $x \lessdot d_1 \lessdot \ldots \lessdot d_m = y$ are two maximal chains between x and y. If $d_1 = c_1$ then $n = m$ by applying the induction hypothesis to the c_1, y-chains. But if $c_1 \neq d_1$ with $x = c_1 \wedge d_1 \lessdot c_1, d_1$ one can conclude that $c_1, d_1 \lessdot c_1 \vee d_1$ by the semimodularity of L.

Then $c_1 \lessdot c_1 \vee d_1 \lessdot \ldots \lessdot y$ is a maximal chain of length $n-1$ and by induction hypothesis every c_1-y-chain is of length $n-1$ and also $d_1 \lessdot c_1 \vee d_1 \ldots \lessdot y$ is a maximal chain of length $n-1$ and again by induction hypothesis every d_1-y-chain is of length $n-1$ and hence $n = m$.

Therefore the existence of a rank function in L is established, so that the semimodular inequality requires a proof.

Let $x \wedge y = c_0 \lessdot \ldots \lessdot c_n = y$ be a maximal $x \wedge y, y$-chain for arbitrary $x, y \in L$. By taking the distinct elements from $x = (x \wedge y) \vee x = c_0 \vee x \leq c_1 \vee x \leq \ldots \leq c_n \vee x = x \vee y$ a maximal $x, x \vee y$-chain is created, which implies $r(y) - r(x \wedge y) \geq r(x \vee y) - r(x)$.

Necessity: Let $x, y \in L$ so that $x \wedge y \lessdot x$ then $r(x \wedge y) = r(x) - 1$ and by using the semimodular inequality one concludes that $r(y) \geq r(x \vee y) - 1$, which implies either the desired result $y \lessdot x \vee y$ or $y = x \vee y$, but this would imply that $x \leq y$ and therefore $x \wedge y = x$, which is a contradiction to the assumption $x \wedge y \lessdot x$. □

As semimodular, point lattices are occuring often another definition is in common use see [2] and [7].

Definition 12. A semimodular, point lattice L is called *geometric*.

At last it is mentioned here that there is a close connection between geometric lattices and matroid theory, but this thesis is not making use of this connection. More information about this topic are found in [2] and [7].

2.3. The incidence algebra

In this chapter the incidence algebra $\Im(P)$ of a poset P is introduced. A few elements of $\Im(P)$ are of special interest in combinatorics, as they allow the counting of chains between two elemens $x, y \in P$ of a given length n in P see also [2, page 138ff.] and [29] for this application in combinatorics.

Thereafter the Möbius function is discussed, that is used to deduce Theorem 2.3.

The incidence algebra

Let P be a poset, then a function $f : P \times P \to \mathbb{C}$ is called an *incidence function* on P if for all $x, y \in P$ with $x \nleq y$ the function $f(x, y)$ vanishes. The set of all incidence functions on P is denoted by $\Im(P)$. Let $f, g \in \Im(P)$ and $\lambda \in \mathbb{C}$ then the pointwise addition $f + g$ is defined by

$$(f + g)(x, y) = f(x, y) + g(x, y) \tag{2.8}$$

and the scalar multiplication λf by

$$(\lambda f)(x, y) = \lambda f(x, y). \tag{2.9}$$

Note that $f + g$ and λf are again elements of $\Im(P)$, so that $\Im(P)$ turns out to be a linear space over the field $\mathbb{C}$. Furthermore the convolution product $f * g \in \Im(P)$ is defined by

$$(f * g)(x, y) = \sum_{x \leq z \leq y} f(x, z) g(z, y) \tag{2.10}$$

with $f * g \in \Im(P)$. The convolution product satisfies

$$f * (g * h) = (f * g) * h \qquad \text{for all } f, g, h \in \Im(P) \tag{2.11}$$

$$(\alpha f) * g = f * (\alpha g) = \alpha(f * g) \qquad \text{for all } f, g \in \Im(P), \alpha \in \mathbb{C}, \tag{2.12}$$

so that $\Im(P)$ becomes an algebra see also [32].

Some incidence functions are especially useful in the view of combinatorial problems. Firstly, the Zeta-function $\zeta \in \Im(P)$

$$\zeta(x, y) = \begin{cases} 1 & x \leq y \\ 0 & \text{else} \end{cases} \tag{2.13}$$

and the Delta-function $\delta \in \Im(P)$

$$\delta(x, y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases} \tag{2.14}$$

are defined.

Secondly, for some $f \in \Im(P)$ there is an inverse incidence function $f^{-1} \in \Im(P)$, which is defined by $f^{-1} * f = \delta$. Observe that f^{-1} exists iff $f(x, x) \neq 0$ for all $x \in P$.

The last important incidence function $\mu \in \mathfrak{I}(P)$, the Möbius function, is then defined as the inverse of the Zeta-function ζ i.e. $\mu = \zeta^{-1}$. The Möbius function can be calculated for a given poset P using the recursion

$$\mu(x,y) = \begin{cases} 1 & x = y \\ -\sum_{x \le z < y} \mu(x,z) & x < y \\ 0 & \text{else} \end{cases}, \tag{2.15}$$

which is derived from the definition of the convolution product in Equation 2.10.

When the Möbius function is known for two posets P and Q, then the Möbius function of the product order $P \times Q$ can be calculated by using the product theorem of the Möbius function see also [32, page 174].

Theorem 13. Let P and Q be posets and μ_P and μ_Q the Möbius function of P and Q, then the Möbius function $\mu_{P\times Q}$ of the product order $P \times Q$ is given by

$$\mu_{P\times Q}((x_1,y_1),(x_2,y_2)) = \mu_P(x_1,x_2)\mu_Q(y_1,y_2) \tag{2.16}$$

if $(x_1,y_1) \le (x_2,y_2)$ in $P \times Q$.

Proof. Let $(x_1,y_1) \le (x_2,y_2)$ then

$$\sum_{(x_1,y_1)\le(z_1,z_2)\le(x_2,y_2)} \mu_{P\times Q}((x_1,y_1),(z_1,z_2)) = \delta_{P\times Q}((x_1,y_1),(x_2,y_2)) = \delta_P(x_1,x_2)\delta_Q(y_1,y_2)$$

implies

$$\sum_{(x_1,y_1)\le(z_1,z_2)\le(x_2,y_2)} \mu_{P\times Q}((x_1,y_1),(z_1,z_2)) = \left(\sum_{x_1\le z_1\le x_2} \mu_P(x_1,z_1)\right)\left(\sum_{y_1\le z_2\le y_2} \mu_Q(y_1,z_2)\right),$$

which proves the claim inductively when using Equation 2.15. □

The Möbius inversion

A versatile tool in combinatorics, the Möbius inversion, utilizes the Möbius function.

Theorem 14. Let P be a poset and $f,g : P :\longrightarrow \mathbb{C}$ be functions on P, then the following inversion principle

$$f(y) = \sum_{x\le y} g(x) \qquad \Longleftrightarrow \qquad g(y) = \sum_{x\le y} \mu(x,y) f(x) \tag{2.17}$$

$$f(y) = \sum_{x\ge y} g(x) \qquad \Longleftrightarrow \qquad g(y) = \sum_{x\ge y} \mu(y,x) f(x) \tag{2.18}$$

is valid.

Proof. Let P be a poset and consider the sum $f(y) = \sum_{x \leq y} g(x) = \sum_{x \in P} g(x)\zeta(x,y)$. Multiplication of this sum with $\mu(y,z)$ for any given $z \geq y$ and summing over all z with $y \leq z$ gives

$$\begin{aligned}\sum_{y \leq z} f(y)\mu(y,z) &= \sum_{y \leq z} \sum_{x \in P} g(x)\zeta(x,y)\mu(y,z) \\ &= \sum_{x \in P} g(x) \sum_{y \leq z} \zeta(x,y)\mu(y,z) \\ &= \sum_{x \in P} g(x)\delta(x,z) = g(z),\end{aligned}$$

so that the first statement is proven. The second statement is established in the same manner. □

Enumerating chains

The Zeta-function ζ of a poset P could be used as a combinatorial tool to calculate chains in the poset P. This becomes very clear, when powers of ζ are considered

$$\zeta^n(x,y) = \sum_{x \leq z_1 \leq \ldots \leq z_{n-1} \leq y} \zeta(x,z_1) \cdots \zeta(z_{n-1},y) \tag{2.19}$$

$$= \sum_{x \leq z_1 \leq \ldots \leq z_{n-1} \leq y} 1, \tag{2.20}$$

that are counting the number of chains in P between x and y of length n if repetitions of the elements are allowed. Likewise the number of chains in P between x and y of length n without repetitions are calculated by introducing the *predecessor function* $\eta = \zeta - \delta$

$$\eta^n(x,y) = \sum_{x < z_1 \leq \ldots \leq z_{n-1} \leq y} \eta(x,z_1) \cdots \eta(z_{n-1},y) \tag{2.21}$$

$$= \sum_{x < z_1 < \ldots < z_{n-1} < y} 1. \tag{2.22}$$

Both results are connected, as stated by the next theorem.

Theorem 15. Let P be a poset and n a non-negative integer. Suppose $\zeta, \eta \in \Im(P)$ are the Zeta-function and the predecessor function of P, respectively. Then the inversion identities

$$\zeta^n = \sum_{i=0}^{n} \binom{n}{i} \eta^i \tag{2.23}$$

$$\eta^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \zeta^i \tag{2.24}$$

hold.

Proof. By the definition of the predecessor function $\zeta = \eta + \delta$ and by taking into account that δ is a commutative and idempotent element in $\Im(P)$ the binomial theorem gives

$$\begin{aligned}\zeta^n &= (\delta + \eta)^n \\ &= \sum_{i=0}^{n} \binom{n}{i} \eta^i \delta^{n-i}\end{aligned}$$

and vice versa

$$\begin{aligned}\eta^n &= (\zeta - \delta)^n \\ &= \sum_{i=0}^{n} \binom{n}{i} \zeta^i (-1)^{n-i} \delta^{n-i},\end{aligned}$$

so that the claim is proven. □

Next it is shown how the number of chains of arbitrary length in a given poset can be calculated if it is assumed that the chains have no repetitions. The proof is adopted from [29].

Theorem 16. Let P be a poset that is not necessarily finite and $x, y \in P$ with $x \leq y$, so that every chain between two elements in P is finite, then the number of chains between x and y without repetitions is given by $(\delta - \eta)^{-1}(x, y) \in \Im(P)$.

Proof. Firstly, observe that the incidence function $\omega := \delta - \eta$ satisfies $\omega(x, x) \neq 0$ for all $x \in P$, so that ω^{-1} exists. Secondly, let m be the maximal length of a chain between x and y, which is finite, implying $\eta^n(u, v) = 0$ for all n with $n > m$ and u, v with $x \leq u \leq v \leq y$. Therefore the total number of chains between x and y is given by

$$\rho(x, y) = (\delta + \eta + \ldots + \eta^m)(x, y).$$

To show that the above expression equals $\omega^{-1}(x, y)$ it is necessary and sufficient that $(\omega\rho)(x, y) = \delta(x, y)$ is satisfied and indeed

$$\begin{aligned}&(\delta - \eta)\left(\delta + \eta + \ldots + \eta^{m+1}\right)(x, y) \\ &= (\delta + \eta + \ldots + \eta^m)(x, y) - \left(\eta + \eta^2 + \ldots + \eta^{m+1}\right)(x, y) \\ &= \left(\delta - \eta^{m+1}\right)(x, y) \\ &= \delta(x, y),\end{aligned}$$

so that the claim is proven. □

3. Combinatorial principles

Before taking a closer look at the partition lattice Π_n it is necessary to recall some combinatorial principles.

Firstly, the Bell numbers and the Stirling numbers of the first and the second kind are introduced and examined, that are occuring frequently in the study of the partition lattice.

Secondly, some graph theoretic concepts are presented that point out the connection between set partitions and graphs.

Then the Inclusion-Exclusion principle is stated and a few applications of it are presented that are utilized in Chapter 4.

In preparation for the next section some remarks about the ring of formal power series are made and the idea of generating functions is exemplified.

In Section 3.5 a brief introduction in the counting of labelled structures and the Set construction is given. The Set construction is a cornerstone when counting chains in the partition lattice in Chapter 4.

Subsequently the partial Bell polynomials arise naturally from the Set operator defined in Section 3.5. The problem of the representation of the iterates of formal power series is then solved by using the notion of the partial Bell polynomials.

The material presented in this chapter was mainly compiled from the sources [13], [34], [8], [24], [17], and [23].

3.1. Set partitions

For the combinatorial study of the partition lattice Π_n in Chapter 4 it is necessary to introduce the notion of a set partition and to recall some elementary combinatorics dealing with set partitions and permutations.

Definition 17. Let X be a finite set then a *set partition* $\pi = \{B_1, \ldots, B_k\}$ of X is a set of nonempty and mutually disjoint subsets B_i of X, so that the union of these sets is X. The elements $B_i \in \pi$ are called the *blocks* of π and the number of blocks of the partition π is denoted by $|\pi|$. The set of all set partitions of X is identified by $\Pi(X)$. If $X = \{1, \ldots, n\}$ is assumed, the notation Π_n is used.

Recall that every set partition $\pi \in \Pi(X)$ induces an equivalence relation $\sim_\pi$ on X and vice versa, by setting

$$x \sim_\pi y :\Leftrightarrow x, y \text{ are in the same block in } \pi \tag{3.1}$$

for all $x, y \in X$. Define by $B_\pi(x)$ the equivalence class of $x \in X$ modulo π by

$$B_\pi(x) = \{y \in X : y \sim_\pi x\} \tag{3.2}$$

or in other words the block of π containing x and the set of all equivalence classes by $X/\sim_\pi$ which equals the set partition π itself.

The *atomic* set partitions $\pi_{x,y} \in \Pi(X)$ with $x, y \in X$ and $x \neq y$ are the set partitions, so that $\{x, y\}$ is a block of $\pi_{x,y}$ and the remaining blocks being singletons.

The first combinatorial question deals with the cardinality $B(n)$ of Π_n, which is called the n-th *Bell number*. The Bell numbers are listed in [28, **A000110**] and obey a recurrence relation.

Theorem 18. For all positive integers n

$$B(n) = \sum_{i=1}^{n} \binom{n-1}{i-1} B(n-i) \tag{3.3}$$

holds with initial condition $B(0) = 1$.

Proof. The proof can be accomplished by using a combinatorial argument. Firstly, Π_n can be partitioned into $n-1$ disjoint sets S_i with $1 \leq i \leq n$, by the block size i of the block containing the distinguished element n. Secondly, the set S_i contains $\binom{n-1}{i-1}B(n-i)$ elements, as there are $\binom{n-1}{i-1}$ possible ways to choose from the set $\{1, \ldots, n-1\}$ the $i-1$ elements, that are sharing the same block with n, and to partition the remaining $n-i$ elements in $B(n-i)$ ways.

The initial condition $B(0) = 1$ makes sense, as there is exactly one possible way to partition the empty set, namely the set partition $\{\}$ containing no block after all. □

Using the above theorem it is an easy task to calculate the first Bell numbers for small values of n. It is observable that $B(n)$ is growing quite rapidly as can be anticipated from Table 3.1. It is possible to give asymptotic estimates for the

n	$B(n)$	n	$B(n)$
1	1	11	678570
2	2	12	4213597
3	5	13	27644437
4	15	14	190899322
5	52	15	1382958545
6	203	16	10480142147
7	877	17	82864869804
8	4140	18	682076806159
9	21147	19	5832742205057

Table 3.1.: Bell numbers $B(n)$

growth of the Bell numbers for large n. The ideas presented in [13, page 522] are leading to the asymptotic estimate

$$B(n) = n! \frac{\exp\{\exp\zeta - 1\}}{\zeta^n \sqrt{2\pi\zeta(\zeta+1)\exp\zeta}} \left(1 + O\left(e^{-\frac{\zeta}{5}}\right)\right), \tag{3.4}$$

where ζ is the solution of the equation $\zeta \exp \zeta = n + 1$, which leads directly to the Lambert W-Function $W(z)$, which is defined by the equation $W(z) \exp W(z) = z$. The properties of the Lambert W-Function are described in [10] and [9], which both gave full asymptotic expansions of it. A method yielding a full asymptotic expansion for the Bell numbers is found in [27].

By counting the set partitions in Π_n, that are containing exactly k blocks, another counting problem is stated. The cardinality of this set is denoted by the numbers $S(n,k)$, which are called the *Stirling numbers of the second kind*. Again it is possible to deduce a recurrence relation for these numbers.

Theorem 19. Let n, k be positive integers then the Stirling numbers of the second kind $S(n,k)$ obey the recurrence relations

$$S(n,k) = kS(n-1,k) + S(n-1,k-1), \tag{3.5}$$

$$S(n,k) = \sum_{i=0}^{n-1} \binom{n-1}{i} S(n-i-1,k-1) \tag{3.6}$$

with the initial conditions $S(n,0) = \delta_{n,0}$ and $S(n,k) = 0$ if $k > n$.

Proof. The subset S_k of all set partitions in Π_n with exactly k blocks can be partitioned into two disjoint sets, where the first one contains the set partitions, so that n is a singleton block, and the second one all set partitions so that n is an element of one of the k blocks. There are $S(n-1,k-1)$ elements in the first set and $kS(n-1,k)$ elements in the second one, as there are $S(n-1,k)$ ways to partition the set $\{1,\dots,n-1\}$ in k blocks and then k possible ways to insert the distinguished element n in one of these k blocks.

The argument for the second claim is similar to the proof of the recurrence relation in Equation 3.3. □

According to the Equation 3.5 the Table 3.2 of the numbers $S(n,k)$ is calculated, which is also found in [28, **A008277**]. Note that the connection between the Bell

n	$S(n,1)$	$S(n,2)$	$S(n,3)$	$S(n,4)$	$S(n,5)$	$S(n,6)$	$S(n,7)$	$S(n,8)$
1	1							
2	1	1						
3	1	3	1					
4	1	7	6	1				
5	1	15	25	10	1			
6	1	31	90	65	15	1		
7	1	63	301	350	140	21	1	
8	1	127	966	1701	1050	266	28	1

Table 3.2.: Triangle of the Stirling numbers of the second kind

numbers and the Stirling numbers of the second kind is

$$B(n) = \sum_{k=0}^{n} S(n,k). \tag{3.7}$$

For $k = 2$ and $k = n - 1$ the values of the $S(n,k)$ are given by

$$S(n,2) = 2^{n-1} - 1, \tag{3.8}$$

$$S(n,n-1) = \binom{n}{2}. \tag{3.9}$$

It is possible to further refine the conditions stipulated on set partitions of an n element set by restricting the block sizes. Let $k_i \geq 0$ be non-negative integers then $S(n, 1^{k_1} \dots n^{k_n})$ denotes the number of set partitions of an n element set, that are containing k_i blocks of size i. Obviously the numbers k_i must fulfill the condition $\sum_{i=1}^n ik_i = n$. The following theorem gives an explicit representation of these numbers.

Theorem 20. Let X be an n element set and $k_1, \dots, k_n$ be non-negative integers, so that $\sum_{i=1}^n ik_i = n$ holds, then

$$S(n, 1^{k_1} \dots n^{k_n}) = \frac{n!}{1!^{k_1} k_1! \cdots n!^{k_n} k_n!} \tag{3.10}$$

is the number of set partitions of n that consists of k_i blocks of size i.

Proof. Firstly, there are

$$\binom{n}{\underbrace{1,\dots,1}_{k_1},\dots,\underbrace{n,\dots,n}_{k_n}} = \frac{n!}{1!^{k_1} \cdots n!^{k_n}}$$

possible ways to choose k_i subsets of size i from an n-element set. Secondly, the order of the blocks is immaterial, so that the k_i subsets of size i can be permuted in $k_i!$ ways, which explains the factor $k_i!$ in the denominator of Equation 3.10 and the result follows. ◻

Hence the following representation of the Stirling numbers of the second kind $S(n,k)$ holds.

Corollary 21. The Stirling numbers of the second kind $S(n,k)$ can be represented by

$$S(n,k) = \sum \frac{n!}{1!^{k_1} k_1! \cdots n!^{k_n} k_n!}. \tag{3.11}$$

The sum ranges over all tuples $(k_1, \dots, k_n)$ satisfying $\sum_{i=1}^n ik_i = n$ and $\sum_{i=1}^n k_i = k$.

Closely related to the Stirling number of the second kind are the Stirling numbers of the first kind $s(n,k)$ and the unsigned Stirling numbers of the first kind $|s(n,k)|$. Before considering the numbers $s(n,k)$ the *unsigned Stirling numbers of the first kind* $|s(n,k)|$ are introduced, as they exhibit a more amenable combinatorial interpretation.

Let X be an n element set and furthermore let $\alpha : X \to X$ be a bijective function, then α is called a *permutation* of the set X and the set of all permutations of X is denoted by $\mathfrak{S}_X$ and in the case $X = \{1, \ldots, n\}$ by $\mathfrak{S}_n$.

While dealing with permutations the notion of a *cylce* occurs. Assume that $i_1, \ldots, i_s$ are distinct elements from the set $\{1, \ldots, n\}$, so that $\alpha(i_j) = i_{j+1}$ and $\alpha(i_s) = i_1$ holds for all $1 \leq j \leq s$, then $(C) = (i_1, \ldots, i_s)$ is called a *cycle* of length s of the permutation α.

The $|s(n, k)|$ are counting the number of permutations of $\mathfrak{S}_n$ with exactly k cycles. Again there is a recurrence relation to calculate the $|s(n, k)|$.

Theorem 22. Let n, k be positive integers then the numbers $|s(n, k)|$ are satisying the recurrence relation

$$|s(n,k)| = |s(n-1,k-1)| + (n-1)|s(n-1,k)|. \tag{3.12}$$

with the initial conditions $|s(n, 0)| = \delta_{n,0}$ and $|s(n, k)| = 0$ for $k > n$.

Proof. First of all check that the initial conditions hold, as only the permutation of the empty set has zero cycles and every permutation must have at least so many elements as it has cycles.

Assume that n is the distinguished element of the permutation $\alpha \in \mathfrak{S}_n$ with α having exactly k cycles, then either n is forming a singleton cycle or the element n must be in one of the k cycles of α. In the first case there are $|s(n-1, k-1)|$ ways to permute the remaining $n - 1$ elements having $k - 1$ cycles and in the second case there are $(n - 1)|s(n - 1, k)|$ ways to firstly permute the $n - 1$ remaining elements with k cycles and secondly to insert the element n in one of the k cycles in $n - 1$ different ways. □

Again there are notable special cases for the numbers $|s(n, k)|$

$$|s(n,1)| = (n-1)!, \tag{3.13}$$

$$|s(n,2)| = H_{n-1}(n-1)! \qquad \text{with } H_n = \sum_{i=1}^{n} \frac{1}{i}, \tag{3.14}$$

$$|s(n,n-1)| = \binom{n}{2}. \tag{3.15}$$

The first values for the $|s(n, k)|$ are listed in Table 3.3 and [28, **A132393**].

Similar to the $S(n, k)$ the number of permutations in $\mathfrak{S}_n$ that have k_i cycles of length i are denoted by $|s(n, 1^{k_1} \ldots n^{k_n})|$.

Theorem 23. Let X be an n element set and let $k_1 \ldots, k_n$ be non-negative integers that are fulfilling $\sum_{i=1}^{n} ik_i = n$ then

$$|s(n, 1^{k_1} \ldots n^{k_n})| = \frac{n!}{1^{k_1}k_1! \cdots n^{k_n}k_n!}$$

holds and furthermore the numbers $|s(n, k)|$ can be represented by

$$|s(n,k)| = \sum \frac{n!}{1^{k_1}k_1! \cdots n^{k_n}k_n!},$$

n	$\lvert s(n,1)\rvert$	$\lvert s(n,2)\rvert$	$\lvert s(n,3)\rvert$	$\lvert s(n,4)\rvert$	$\lvert s(n,5)\rvert$	$\lvert s(n,6)\rvert$	$\lvert s(n,7)\rvert$	$\lvert s(n,8)\rvert$
1	1							
2	1	1						
3	2	3	1					
4	6	11	6	1				
5	24	50	35	10	1			
6	120	274	225	85	15	1		
7	720	1764	1624	735	175	21	1	
8	5040	13068	13132	6769	1960	322	28	1

Table 3.3.: Triangle of the unsinged Stirling numbers of the first kind

where the sum ranges over all n tuples $(k_1, \dots, k_n)$, so that the constraints $\sum_{i=1}^{n} k_i = k$ and $\sum_{i=1}^{n} ik_i = n$ are satisfied.

Proof. Firstly, recall the Theorem 20, which is utilized in the proof of this theorem. Secondly, assume that $\pi \in \Pi_n$ is a set partition that contains exactly k_i blocks of size i, then every block in π of size i gives rise to exactly $(i-1)!$ different cyclic arrangements and therefore

$$\begin{aligned} |s(n, 1^{k_1} \dots n^{k_n})| &= S(n, 1^{k_1} \dots n^{k_n}) \prod_{i=1}^{n} (i-1)!^{k_i} \\ &= \frac{n!}{1^{k_1}k_1! \cdots n^{k_n}k_n!}, \end{aligned}$$

holds, which proves the desired claim. □

Another useful application of the Stirling numbers of the first and the second kind occurs in the conversion from falling factorials $z^{\underline{n}}$ into normal powers z^n and vice versa. Firstly, recall that the n-th falling factorial of z is defined for all positive integers n by

$$z^{\underline{n}} = z(z-1)\cdots(z-n+1) \tag{3.16}$$

and $z^{\underline{0}} = 1$ is stipulated for $n = 0$ and secondly, that the remarkable difference formula

$$(z+1)^{\underline{n}} - z^{\underline{n}} = nz^{\underline{n-1}} \tag{3.17}$$

is valid, which is reminiscent of the derivative of the function z^n. Furthermore define the n-th rising factorial of z for all positive integers n by

$$z^{\overline{n}} = z(z+1)\cdots(z+n-1) \tag{3.18}$$

and likewise set $z^{\overline{0}} = 1$. Observe the conversion between the rising and falling factorial by

$$z^{\overline{n}} = (-1)^n(-z)^{\underline{n}} \qquad\qquad z^{\underline{n}} = (-1)^n(-z)^{\overline{n}}. \tag{3.19}$$

The next theorem reveals the link between the z^n and $z^{\underline{n}}$ by employing the unsigned Stirling numbers of the first kind and the Stirling numbers of the second kind.

Theorem 24. Let n be a non-negative integer then the conversion formulas

$$z^n = \sum_{k=0}^{n} S(n,k) z^{\underline{k}}, \tag{3.20}$$

$$z^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k} |s(n,k)| z^k \tag{3.21}$$

are valid.

Proof. The proof proceeds with induction over n. Firstly, for $n = 0$ the claim is true for both equations. Secondly, assume that the claim is true for all $n' \leq n+1$. Multiplication of the first equation with z and the second one by $z - n$ yields

$$\begin{aligned} z^{n+1} &= \sum_{k=0}^{n} S(n,k)(kz^{\underline{k}} + z^{\underline{k+1}}) \\ &= \sum_{k=1}^{n+1} (kS(n,k) + S(n,k-1)) z^{\underline{k}} \\ &= \sum_{k=0}^{n+1} S(n+1,k) z^{\underline{k}} \end{aligned}$$

and

$$\begin{aligned} z^{\underline{n+1}} &= \sum_{k=0}^{n} (-1)^{n-k} |s(n,k)| (z^{k+1} - nz^k) \\ &= \sum_{k=1}^{n+1} (-1)^{n+1-k} \left(|s(n,k-1)| + n|s(n,k)| \right) z^k \\ &= \sum_{k=0}^{n+1} (-1)^{n+1-k} |s(n+1,k)|, \end{aligned}$$

so that the claim is true for all n. Note the subtle change in the lower bound of the summation from $k = 0$ to $k = 1$ and back to $k = 0$, which accounts for the initial conditions $|s(n,0)|, S(n,0) = \delta_{n,0}$. □

The conversion numbers $s(n,k) = (-1)^{n-k}|s(n,k)|$ are occurring frequently and are called the *Stirling numbers of the first kind*. The Stirling numbers of the first and the second kind obey an inversion relation.

Theorem 25. Let $S(n,k)$ and $s(n,k)$ denote the Stirling numbers of the second and the first kind then for all non-negative integers n and m the equations

$$\sum_{k=0}^{n} S(n,k) s(k,m) = \delta_{n,m}, \tag{3.22}$$

$$\sum_{k=0}^{n} s(n,k) S(k,m) = \delta_{n,m} \tag{3.23}$$

hold, where $\delta_{n,m}$ is 0 whenever $n \neq m$ and 1 if $n = m$.

Proof. The proof is established by the use of Theorem 24, as $z^n = \sum_{k=0}^{n} S(n,k) z^{\underline{k}}$ and $z^{\underline{k}} = \sum_{m=0}^{k} s(k,m) z^m$ yields

$$z^n = \sum_{m=0}^{n} \left(\sum_{k=m}^{n} S(n,k) s(k,m) \right) z^m.$$

Comparing the coefficients of the powers z^k yields the desired result. The second equation is proven analogous. □

3.2. Graph theoretic concepts

It turns out that many questions dealing with set partitions can be translated into the language of graph theory and vice versa. Therefore a few basic graph theoretic concepts are compiled in this brief section. Most of the definitions are found in [17, Chapter 11].

Firstly, , the notion of a simple, undirected graph is given.

Definition 26. A *simple, undirected graph* or short graph G is a pair $G(V,E)$ consisting of a finite set V and a subset E of the two element subsets $[V]^2$ of V. The elements of V are called the *vertices* and the elements in E the *edges*. It is said that u and v are *adjacent* in G if $\{u,v\} \in E$ holds. It is convenient to represent a graph G in a pictorial way like exemplified in Figure 3.1.

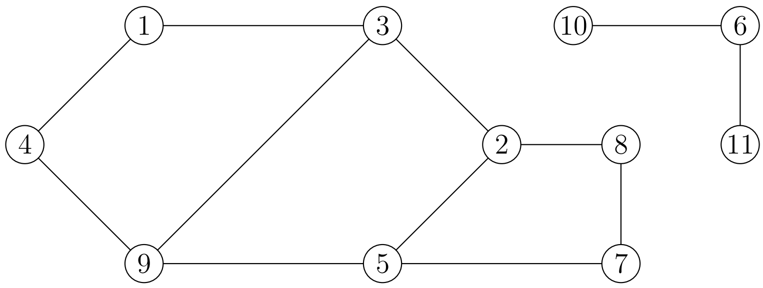

Figure 3.1.: A pictorial representation of a graph

Some special graphs are useful in later chapters. Denote by K_n the *complete graph* with $|V| = n$ and $E = [V]^2$ and by $\overline{K}_n$ the *empty graph* with $|V| = n$ and $E = \emptyset$.

Secondly, the concepts of components, connectedness, subgraphs, induced subgraphs and spanning subgraphs are discussed in order to depict the similarities between set partitions, supremum operation and graphs in Chapter 4.

Definition 27. Let $G(V,E)$ be a graph then $x, y \in V$ are *connected* in G if there is a sequence $e_1, \ldots, e_n$ of edges with $e_i \in E$ satisfying $x \in e_1, y \in e_n$ and $e_i \cap e_{i+1} \neq \emptyset$ for all $1 \leq i \leq n-1$. By introducing the notation $x \equiv_G y$ if x and y are connected in G it is been readily checked that $\equiv_G$ is an equivalence relation on V. The equivalence classes $(v) \in V/{\equiv_G}$ for all $v \in V$ are called the *component sets* of G. If $V/{\equiv}$ contains only one component set, then G is a *connected graph*. The component sets of the graph depicted in Figure 3.1 are $\{1,2,3,4,5,7,8,9\}$ and $\{6,10,11\}$, respectively.

Let $G(V,E)$ be a graph and $U \subseteq V$ and $F \subseteq E$, so that for all $\{u,v\} \in F$ it is $u,v \in U$ then $H(U,F)$ is called a *subgraph* of G. If $U \subseteq V$ and if for all $u,v \in U$, it is $\{u,v\} \in F$ iff $\{u,v\} \in E$, then $H[U] := H(U,F)$ is called the subgraph of G *induced* by U. Let $v \in V$ and (v) the component set belonging to v then the by (v) induced subgraph $G[(v)]$ is called a *component* of G. A subgraph $H(U,F)$ of a graph $G(V,E)$ is called a *spanning subgraph* of G if $U = V$.

Another graph theoretic concept arises by introducing the notion of admissible colorings of a graph.

Definition 28. A function $\phi : V \to \{1,\dots,\lambda\}$ that assigns every vertex of a graph $G(V,E)$ a color from the λ element set $\{1,\dots,\lambda\}$ is called a *coloring* of G. A coloring ϕ of G is called *admissible* if adjacent vertices $u,v \in V$ satisfy $\phi(u) \neq \phi(v)$.

The smallest number λ, such that an admissible coloring of G exists, is called the *chromatic number* $\chi(G)$ of G. The number of possible colorings with at most λ colors is given by the *chromatic polynomial* $\chi(G,\lambda)$.

Note that the chromatic polynomial of the complete graph K_n is given by $\chi(K_n,\lambda) = \lambda^{\underline{n}}$ and of the empty graph $\overline{K}_n$ by $\chi(\overline{K}_n,\lambda) = \lambda^n$.

Closely, related to the notion of admissible colorings is the set of independent partitions Π_G of a graph G, which leads to the independent set polynomial $M(G,\lambda)$.

Definition 29. Let $G(V,E)$ be a graph. A partition $\pi \in \Pi(V)$ is called *independent* in G if $x \sim_\pi y$ implies that x and y are not adjacent in G. The set of all independent partitions in G is denoted by Π_G. The independent set polynomial $M(G,\lambda)$ is defined by

$$M(G,\lambda) = \sum_{\pi \in \Pi_G} \lambda^{|\pi|}. \tag{3.24}$$

The set of independent partitions Π_G of G allows a representation of the chromatic polynomial $\chi(G,\lambda)$ of G see also [33].

Theorem 30. Let $G(V,E)$ be a graph G, then

$$\chi(G,\lambda) = \sum_{\pi \in \Pi_G} \lambda^{\underline{|\pi|}} \tag{3.25}$$

holds.

Proof. Every admissible coloring of G can be identified with an independent partition π of G, where the vertices in one block of π are bearing the same color. Every independent partition $\pi \in \Pi_G$ contributes $\lambda^{\underline{|\pi|}}$ admissible coloring of G with λ colors, as the first block of π can be colored with λ colors, the second block with the remaining $(\lambda - 1)$ colors and the last block with $(\lambda - |\pi| + 1)$ colors. $\square$

Let $G(V, E)$ be the example graph in 3.1 then the chromatic polynomial $\chi(G, \lambda)$ and the independent set polynomial $M(G, \lambda)$ are

$$\begin{aligned}\chi(G, \lambda) = &\lambda^{11} - 12\lambda^{10} + 66\lambda^9 - 217\lambda^8 + 468\lambda^7 \\ &- 684\lambda^6 + 675\lambda^5 - 432\lambda^4 + 162\lambda^3 - 27\lambda^2\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}M(G, \lambda) = &\lambda^{11} + 43\lambda^{10} + 681\lambda^9 + 5039\lambda^8 + 18311\lambda^7 \\ &+ 31621\lambda^6 + 23281\lambda^5 + 5851\lambda^4 + 322\lambda^3 + 2\lambda^2 .\end{aligned} \tag{3.27}$$

The independent set polynomial $M(G, \lambda)$ and the chromatic polynomial can be transformed into each other by employing the Stirling numbers of the first and the second kind.

Theorem 31. Let $G(V, E)$ be a graph then

$$T\left[\chi(G, \lambda)\right] = M(G, \lambda) \tag{3.28}$$

$$T^{-1}\left[M(G, \lambda)\right] = \chi(G, \lambda), \tag{3.29}$$

where T and T^{-1} denotes the linear operators

$$T : \mathbb{C}[\lambda] \to \mathbb{C}[\lambda] : \sum_i a_i \lambda^i \to \sum_i a_i \sum_{k=0}^{i} S(i, k)\lambda^k \tag{3.30}$$

$$T^{-1} : \mathbb{C}[\lambda] \to \mathbb{C}[\lambda] : \sum_i a_i \lambda^i \to \sum_i a_i \sum_{k=0}^{i} s(i, k)\lambda^k. \tag{3.31}$$

Proof. The proof is only presented for the operator T and makes use of Theorem 25. Assume that $\chi(G, \lambda) = \sum_{\pi \in \Pi_G} \lambda^{\underline{\pi}}$ is the chromatic polynomial of the graph G, then application of T to $\chi(G, \lambda)$ leads to

$$\begin{aligned}T\left[\chi(G, \lambda)\right] &= \sum_{\pi \in \Pi_G} T\left[\lambda^{\underline{\pi}}\right] \\ &= \sum_{\pi \in \Pi_G} T\left[\sum_{i=0}^{|\pi|} s(|\pi|, i)\lambda^i\right] \\ &= \sum_{\pi \in \Pi_G} \sum_{k=0}^{i} \left(\sum_{i=0}^{|\pi|} s(|\pi|, i) S(i, k)\right) \lambda^k \\ &= \sum_{\pi \in \Pi_G} \lambda^{|\pi|} = M(G, \lambda).\end{aligned}$$

The second equation can be deduced by a similar argument. $\square$

3.3. Inclusion-Exclusion principle

The Inclusion-Exclusion principle allows to count the elements of the union of not necessarily disjoint, finite sets S_i by successively adding and subtracting terms. A very elegant proof of this principle was presented to the author during a lecture [11] by exploiting only elementary combinatorial observations.

The first observation is stated in the Lemma 32.

Lemma 32. Let I be a nonempty set. Then the number of subsets of I with an even number of elements equals the number of subsets of I with an odd number of elements, so that

$$\sum_{J \subseteq I} (-1)^{|J|} = 0 \tag{3.32}$$

holds.

Proof. In order to show that two sets A and B have the same number of elements it suffices to find a bijective function $f\colon A \to B$.

Let therefore $\mathcal{E}$ be the set of subsets of I with an even number of elements and $\mathcal{O}$ be the set of subsets of I with an odd number of elements. Let $x \in I \neq \emptyset$ be an arbitrary element and define $f\colon \mathcal{E} \to \mathcal{O}$ by

$$f(J) = \begin{cases} J \cup \{x\} & x \notin J \\ J \setminus \{x\} & x \in J \end{cases} \tag{3.33}$$

for all $J \in \mathcal{E}$, then f is obviously the desired bijective function, which proves the claim. □

Utilizing Lemma 32 the Inclusion-Exclusion principle can be stated and proven in a very brief and concise way.

Theorem 33. Let $S_1, \ldots, S_n$ be finite sets, then

$$\left| \bigcup_{i=1}^{n} S_i \right| = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1} \left| \bigcap_{i \in I} S_i \right| \tag{3.34}$$

holds.

Proof. Let $a \in \bigcup_{i=1}^{n} S_i = S$ and $J(a) := \{i \in \{1, \ldots, n\} : a \in S_i\}$, then $J(a) \neq \emptyset$ for all $a \in S$. The left-hand side of the Equation 3.34 counts the number of elements in S or in other words $\sum_{a \in S} 1$. Due to Lemma 32 it is $1 = \sum_{I \subseteq J(a), I \neq \emptyset} (-1)^{|I|-1}$ as $J(a) \neq \emptyset$, implying

$$\begin{aligned} \sum_{a \in S} 1 &= \sum_{a \in S} \sum_{I \subseteq J(a), I \neq \emptyset} (-1)^{|I|-1} \\ &= \sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|-1} \sum_{a : I \subseteq J(a)} 1. \end{aligned}$$

For all $I \subseteq \{1, \ldots, n\}$ it is true that $a \in \bigcap_{i \in I} S_i$ iff $I \subseteq J(a)$, so that

$$\left|\bigcup_{i=1}^{n} S_i\right| = \sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|-1} \left|\bigcap_{i \in I} S_i\right|$$

and the claim follows. □

The Inclusion-Exclusion principle can also be stated in the form

$$\left|\bigcap_{i=1}^{n} \overline{S_i}\right| = \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \left|\bigcap_{i \in I} S_i\right|, \tag{3.35}$$

which can be readily achieved by $\bigcap_{i=1}^{n} \overline{S_i} = \overline{\bigcup_{i=1}^{n} S_i} = S \setminus \bigcup_{i=1}^{n} S_i$.

In many applications of the Inclusion-Exclusion principle the sum occurring on the right-hand side of Equation 3.35 can be simplified, namely if the size of the set $\bigcap_{i \in I} S_i$ depends only on the size $k = |I|$ of the index set I. The sum then assumes the concise form

$$\left|\bigcap_{i=1}^{n} \overline{S_i}\right| = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(k), \tag{3.36}$$

if $f(k)$ is stipulated as the size of the set $\bigcap_{i \in I} S_i$ for all I with $|I| = k$. Observe that the factor $\binom{n}{k}$ occurs, as there are $\binom{n}{k}$ ways to select k of the n sets $S_1, \ldots, S_n$.

There is a vast amount of counting problems that can be solved by using the Inclusion-Exclusion principle. Consider for example the problem to count the set $\mathcal{F}(n,k)$ of all surjective functions $f : R \to T$, where R and T are finite sets with $|R| = n$ and $|T| = k$.

This problem can be solved by defining the set of functions

$$S_i = \{f : R \to T | f(x) \neq i, \text{ for all } x \in R\} \tag{3.37}$$

with $1 \leq i \leq k$. Then $F(n,k) = |\mathcal{F}(n,k)|$ equals $|\bigcap_{i=1}^{k} \overline{S_i}|$, which follows from the definition of a function to be onto. By sorting the sum in the Inclusion-Exclusion principle by the number of elements in the subsets I the representation

$$F(n,k) = \sum_{j=0}^{k} (-1)^j \sum_{I \subseteq \{1,\ldots,k\}, |I| = j} \left|\bigcap_{i \in I} S_i\right| \tag{3.38}$$

is derived and furthermore the inner sum only depends on j resembling Equation 3.36, as it counts the number of functions $f : R \to T$, so that at least j elements from T are not in the range of f. The inner sum can be simplified. Firstly, j among the k elements of T are selected which can be accomplished in $\binom{k}{j}$ different ways and secondly, there are $(k-j)^n$ mappings from an n element set to an $k-j$ element. This leads to the final representation

$$F(n,k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n \tag{3.39}$$

for the $F(n, k)$.

A very nice application of the Inclusion-Exclusion principle yields an interesting and useful representation of the chromatic polynomial $\chi(G, \lambda)$ of the graph G. This result was found by Whitney [33, theorem 12.5.].

Theorem 34. Let $G(V, E)$ be a graph with n vertices and m edges. Then the chromatic polynomial $\chi(G, \lambda)$ can be expressed by

$$\chi(G, \lambda) = \sum_{i=0}^{m} \sum_{j=0}^{n} (-1)^i c_{i,j}(G) \lambda^j, \tag{3.40}$$

where $c_{i,j}(G)$ denotes the number of spanning subgraphs of G that are containing i edges and j components.

Proof. Let S_e for all $e \in E$ the set of all colorings of G with λ colors, so that the edge e is monochromatic. Then by the Inclusion-Exlcusion principle

$$\begin{aligned} \chi(G, \lambda) &= \left| \bigcap_{e \in E} \overline{S}_e \right| = \sum_{F \subseteq E} (-1)^{|F|} \left| \bigcap_{e \in F} S_e \right| \\ &= \sum_{i=0}^{m} (-1)^i \left\{ \sum_{F \subseteq E, |F| = i} \left| \bigcap_{e \in F} S_e \right| \right\} \end{aligned}$$

follows.

The sum in braces counts the number of colorings with λ colors of the spanning subgraphs of G with exactly i edges, so that all edges in F are monochromatic. This implies that all vertices in G that are in the same component of this spanning subgraph must be monochromatic. If the spanning subgraph contains exactly j components it can be colored in λ^j possible ways and the result follows. □

Theorem 34 could be used to establish a combinatorial identity that is seemingly hard to proof by an easy combinatorial argument. Assume that G is chosen to be the complete graph K_n then $\chi(K_n, \lambda) = \lambda^{\underline{n}}$, so that

$$\lambda^{\underline{n}} = \sum_{j=1}^{n} \left(c^e_{n,j} - c^o_{n,j} \right) \lambda^j \tag{3.41}$$

holds, where $c^e_{n,j}$ denotes the number of graphs with n vertices and j components with an even number of edges and likewise $c^o_{n,j}$ denotes the number of graphs with n vertices and j components having an odd number of edges.

By comparing the powers of λ and utilizing Theorem 24 the interesting correspondence

$$s(n, j) = c^e_{n,j} - c^o_{n,j} \tag{3.42}$$

is derived, which is seemingly hard to prove by basic combinatorial arguments.

Finally observe that Theorem 34 implies that the independent set polynomial $M(G, \lambda)$ can be represented in the form

$$M(G, \lambda) = \sum_{j=0}^{n} \left(c_j^e(G) - c_j^o(G)\right) \sum_{i=0}^{j} S(j, i)\lambda^i, \tag{3.43}$$

where $c_j^e(G)$ and $c_j^o(G)$ are counting the spanning subgraphs of G with j components having an even or odd number of edges.

The last application of the Inclusion-Exclusion principle yields an alternative proof for Theorem 15.

Proof: 2. Let $x, y \in P$ and $A_{n,k}$ be the set of all chains $z_0 = x \leq z_1 \leq \ldots \leq z_{n-1} \leq y = z_n$ of length n with k repetitions, so that there are $1 \leq i_1 < \ldots < i_k \leq n$ with $z_{i_j - 1} = z_{i_j}$. Observe that $\zeta^n(x, y) = \sum_{k=0}^{n} |A_{n,k}|$ holds. Furthermore let B_i be the set of all chains $z_0 = x < z_1 < \ldots < z_{i-1} < y = z_i$ of length i with no repetitions, so that $\eta^i(x, y) = |B_i|$ holds. Every chain in $A_{n,k}$ arises from a chain in B_{n-k} by inserting the repetitions in $\binom{n}{n-k}$ different places, which proves the first claim.

To prove the second assertion the sets S_i for $i = 1, \ldots, n$ are introduced containing the chains $z_0 = x \leq z_1 \ldots \leq z_{n-1} \leq y = z_n$ so that $z_{i-1} = z_i$ holds. Observe that B_n equals $\bigcap_{i=1}^{n} \overline{S}_i$. Hence the Inclusion-Exclusion principle implies

$$|B_n| = \sum_{I \subseteq \{1, \ldots, n\}} (-1)^{|I|} \left| \bigcap_{i \in I} S_i \right|.$$

In this case the summands on the right-hand side are only dependent on the size of I. Hence

$$|B_n| = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{n-k} \zeta^k(x, y)$$

follows, proving the second claim. □

3.4. Formal power series

In Section 3.5 formal power series are used to answer combinatorial questions, therefore the most important definitions and theorems about formal power series are stated here. At the end a small example demonstrating the usefulness of formal power series as generating functions is given. See also [19], [17] and, [34] for a multitude of examples of this method. A more thorough introduction into formal power series and its applications is found in [3, Chapter 2], [34, Chapter 2], [15], and [19].

First the ring of formal power series $(\mathbb{C}[[z]], +, \cdot)$ over the field $\mathbb{C}$ is defined.

Definition 35. Let $\mathbb{C}[[z]]$ be the set of all infinite sequences $\{a_n\}_{n\geq 0}$ endowed with the operations $+$ and $\cdot$ defined by

$$\{a_n\} + \{b_n\} = \{a_n + b_n\} \tag{3.44}$$

$$\{a_n\} \cdot \{b_n\} = \left\{ \sum_{k=0}^{n} a_k b_{n-k} \right\}, \tag{3.45}$$

then $\mathbb{C}[[z]]$ is a commutative ring. By identifying each element $\{a_n\} \in \mathbb{C}[[z]]$ with the formal power series in the indeterminate z

$$A(z) = \sum_{n\geq 0} a_n z^n \tag{3.46}$$

the operations $+$ and $\cdot$ are naturally defined as the sum $A(z)+B(z)$ and the product $A(z)B(z)$ of the two formal power series $A(z)$ and $B(z)$.

Observe that questions of convergence are immaterial when dealing with formal power series. Therefore the following power series $G(z) \in \mathbb{C}[[z]]$

$$G(z) = \sum_{n\geq 0} \frac{2^{\binom{n}{2}}}{n!} z^n \tag{3.47}$$

appearing in Equation 3.80 is a well-defined object.

Definition 36. Let $A(z) \in \mathbb{C}[[z]]$ and define the *coefficient extraction operator* by

$$[z^n]\, A(z) := a_n. \tag{3.48}$$

It is demanded that the operator satisfies the law

$$\beta\, [z^n]\, A(z) = \left[\frac{z^n}{\beta}\right] A(z) \tag{3.49}$$

for all $\beta \in \mathbb{C}$, which proves convenient in Section 3.5, when dealing with exponential generating functions by setting $\beta = n!$.

Let $F(z) \in \mathbb{C}[[z]]$, if there exists a $G(z) \in \mathbb{C}[[z]]$ such that $F(z)G(z) = 1$, then $G(z)$ is the *inverse* of $F(z)$. It is easily seen that $G(z)$ uniquely exists iff $f_0 \neq 0$. Furthermore the operation of *composition* of formal power series is defined by $(F \circ G)(z) = F(G(z))$. It can be shown that the composition only makes sense if $F(z)$ is a polynomial or $g_0 = 0$ holds. If $F(G(z)) = G(F(z)) = z$ is satisfied then $G(z)$ is called the *compositional inverse* of $F(z)$, see also [34] and [32] for a rigorous introduction into this topic.

The formal power series $E(z), L(z) \in \mathbb{C}[[z]]$

$$E(z) = \sum_{n\geq 1} \frac{z^n}{n!} \tag{3.50}$$

$$L(z) = \sum_{n\geq 1} (-1)^{n-1} \frac{z^n}{n} \tag{3.51}$$

appear in Section 3.5 and for convenience $E(z) = \exp z - 1$, $L(z) = \ln(1+z)$ is set. Note that $E(z)$ and $L(z)$ are compositional inverses of each other. This can be easily shown by utilizing the tools developed in Section 3.7.

Formal power series are a quite versatile tool in combinatorics, as they can be used as generating functions of sequences see [34], [19], and [17] for applications of this method in combinatorics. The idea of generating functions is to compress the information of a sequence $\{a_n\}_{n\geq 0}$, where each a_n counts the number of a certain combinatorial structure of the size n, in a formal power series $A(z) = \sum_{n\geq 0} a_n z^n$.

To exemplify the use of generating functions in combinatorics see [34, page 16], the ordinary generating function $A_k(z) = \sum_{n\geq 0} S(n,k) z^n$ for the sequence $\{S(n,k)\}_{n\geq 0}$ is determined for a fixed integer $k \geq 1$. In order to determine $A_k(z)$ the recurrence relation 3.5 of the $S(n,k)$ with $n, k \geq 1$ is multiplied by z^n and then this expression is summed over all $n \geq 1$ yielding

$$\sum_{n\geq 1} S(n,k) z^n = z \sum_{n\geq 1} S(n-1,k-1) z^{n-1} + kz \sum_{n\geq 1} S(n-1,k) z^{n-1}, \tag{3.52}$$

so that

$$\frac{1}{z}\left(A_k(z) - S(0,k)\right) = A_{k-1}(z) + kA_k(z) \tag{3.53}$$

$$\frac{1}{z} A_k(z) = A_{k-1}(z) + kA_k(z) \tag{3.54}$$

if $S(0,k) = \delta_{0,k}$ is accounted for. Hence the recurrence relation for the $A_k(z)$

$$A_k(z) = \frac{z}{1-zk} A_{k-1}(z) \qquad \text{for all } k \geq 1 \tag{3.55}$$

is derived and by employing the initial condition $A_0(z) = 1$

$$A_k(z) = z^k \prod_{i=1}^{k} \frac{1}{1-iz} \tag{3.56}$$

becomes the desired generating function which can be further modified by expanding the product via a partial fraction decomposition

$$A_k(z) = z^k \sum_{i=1}^{k} \frac{\gamma_i}{1-iz} \tag{3.57}$$

with

$$\gamma_i = (-1)^{k-i} \frac{i^{k-1}}{(i-1)!(k-i)!} \tag{3.58}$$

implying

$$S(n,k) = [z^n] A_k(z) = [z^{n-k}] \sum_{i=1}^{k} \frac{\gamma_i}{1-iz} = \sum_{i=1}^{k} \gamma_i i^{n-k} \tag{3.59}$$

and finally

$$S(n,k) = \sum_{i=1}^{k} (-1)^{k-i} \frac{i^n}{i!(k-i)!} \tag{3.60}$$

$$= \frac{1}{k!} \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} i^n \tag{3.61}$$

a closed form for the $S(n,k)$ is found, that is valid for all non-negative n and positive k.

In comparision with Equation 3.39 the relationship $k!S(n,k) = F(n,k)$ is deduced.

3.5. The counting of labelled structures

This section represents a brief introduction into the counting of labelled structures, as presented in [13]. Cornerstone of this chapter is the notion of a labelled class and the construction of new labelled classes from already existing ones by the means of the operations sum, labelled product, sequence, set, and cycle construction. The notation used in this section is taken from [13]. Other introductions with different viewpoints are found in [4], [34] , and [3].

The framework developed in this section allows the specifaction of a multitude of structures found in the partition lattice and is used in Chapter 4. In a natural way the set construction turns out to be of the greatest usefulness when counting structures in Π_n.

Firstly, the notion of a labelled class is given following the ideas presented in [13].

Definition 37. A combinatorial class $(\mathcal{A}, |\cdot|)$ is a at most countable set $\mathcal{A}$ endowed with a size function $|\cdot|\colon \mathcal{A} \to \mathbb{N} \cup \{0\}$, which assign every object $\alpha \in \mathcal{A}$ a non-negative integer $|\alpha|$, so that the set of objects of a given size n in $\mathcal{A}$ is finite. Subsequently the shorthand form $\mathcal{A}$ is used to denote the combinatorial class $(\mathcal{A}, |\cdot|)$ if the definition of the size function is unambiguous.

It is assumed that every object of size n consists of n smaller objects called the *atoms* of the object and that these atoms are labelled with positive integers. Such an object is called a *labelled object* of size n. A labelled object of size n labelled with the integers $\{1, \ldots, n\}$ is called a *well-labelled* object. A combinatorial class $(\mathcal{A}, |\cdot|)$ of well-labelled objects is called a *labelled class*.

Two trivial examples for labelled classes are the *atomic labelled class* $\mathcal{Z}$, which contains only one object of size 1, and the *empty labelled class* $\mathcal{E}$ containing no object at all. Although being trivial examples, these two classes are often serving as fundamental building blocks of more complicated labelled classes.

The *exponential generating function* $\hat{A}(z)$ of a labelled class $\mathcal{A}$ is defined by

$$\hat{A}(z) = \sum_{\alpha \in \mathcal{A}} \frac{z^{|\alpha|}}{|\alpha|!} = \sum_{n \geq 0} a_n \frac{z^n}{n!}, \tag{3.62}$$

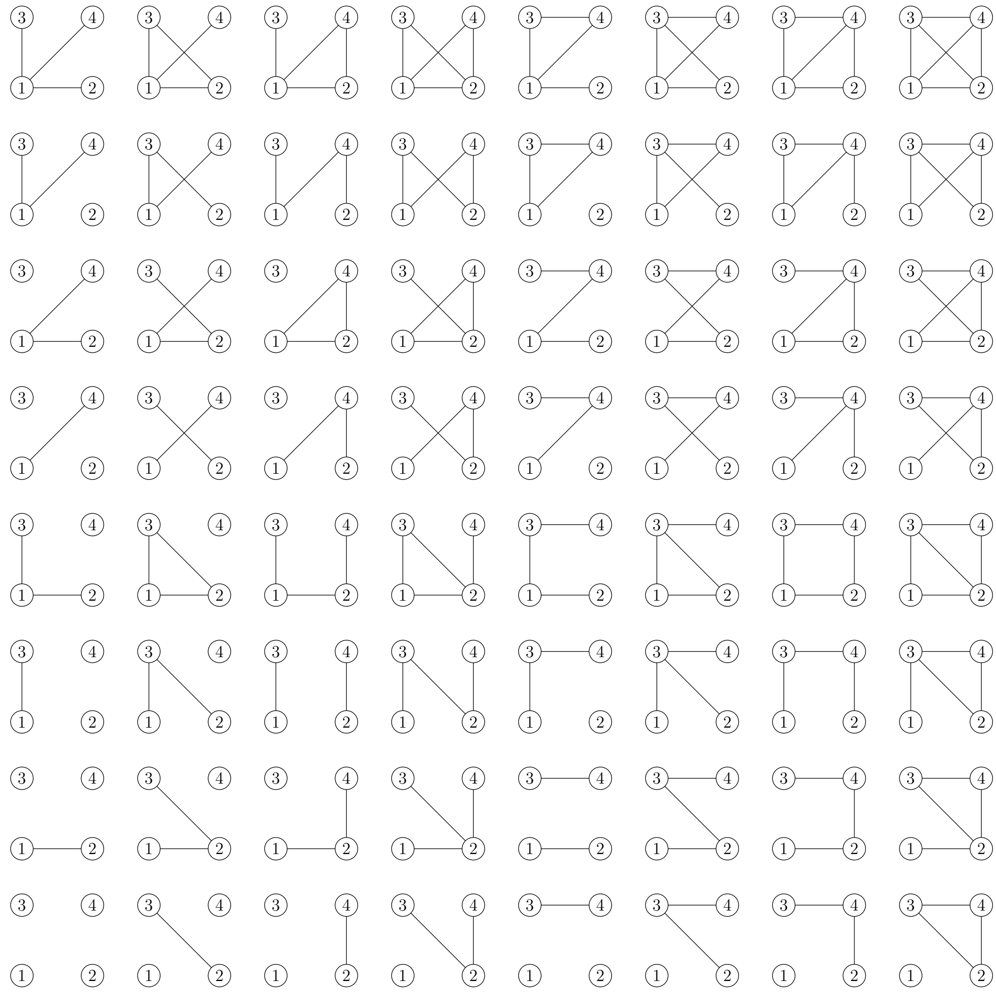

Figure 3.2.: The labelled class of all graphs $G \in \mathcal{G}$ with size four.

where a_n denotes the number of objects in $\mathcal{A}$ of size n.

The first trivial examples of exponential generating functions $\hat{Z}(z) = z$ and $\hat{E}(z) = 1$ are mentioned, being the exponential generating functions of the atomic $\mathcal{Z}$ and the empty $\mathcal{E}$ labelled class.

The above definition is quite broad and better understandable by giving concrete examples. Consider the set of graphs $\mathcal{G}$, where for each $G(V, E) \in \mathcal{G}$ the size of $G(V, E)$ is determined by the number of vertices of V. Then $\mathcal{G}$ becomes a combinatorial class. If furthermore the n vertices of any given graph of size n are labelled with the labels $\{1, \ldots, n\}$, then $\mathcal{G}$ becomes a labelled class. Figure 3.2 depicts the labelled class of all graphs with four vertices, which contains $2^{\binom{4}{2}} = 64$ elements. The exponential generating function $\hat{G}(z)$ of $\mathcal{G}$ is then given by

$$\hat{G}(z) = \sum_{n \geq 0} 2^{\binom{n}{2}} \frac{z^n}{n!}. \tag{3.63}$$

Given two disjoint labelled classes $(\mathcal{A}, |\cdot|_{\mathcal{A}})$ and $(\mathcal{B}, |\cdot|_{\mathcal{B}})$ the *sum* $\mathcal{C} = \mathcal{A} + \mathcal{B}$ of both classes is defined by $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$. Where for every $\gamma \in \mathcal{C}$ the size $|\gamma|$ is given

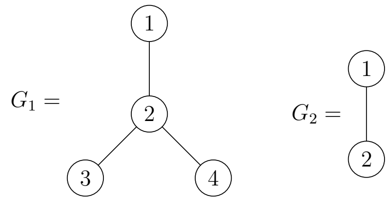

Figure 3.3.: The well-labelled graphs G_1 and G_2

by

$$|\gamma|_C = \begin{cases} |\gamma|_{\mathcal{A}} & \gamma \in \mathcal{A} \\ |\gamma|_{\mathcal{B}} & \gamma \in \mathcal{B} \end{cases}. \tag{3.64}$$

It can be readily concluded that $c_n = a_n + b_n$ and $\hat{C}(z) = \hat{A}(z) + \hat{B}(z)$ holds, respectively. If the labelled classes $\mathcal{A}$ and $\mathcal{B}$ are not disjoint, it is still possible to define the sum $\mathcal{A} + \mathcal{B}$ by setting $C = \mathcal{E}_1 \times \mathcal{A} + \mathcal{E}_2 \times \mathcal{B}$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are disjoint empty labelled classes.

Given two labelled classes $\mathcal{A}$ and $\mathcal{B}$ the construction of the *labelled product* $\mathcal{A} \star \mathcal{B}$ is defined. The labelled product leads to a rich class of new constructions, as presented in [13], where the focus here lies mainly on the set construction.

In order to understand the labelled product the *expansion* of a well-labelled object $\alpha \in \mathcal{A}$ is introduced. An expansion of the well-labelled object α of size n is an injective, order preserving mapping $\rho : \{1, \ldots, n\} \to \mathbb{Z}$, that conveys a relabelling $\rho(\alpha)$ of the labels of the well-labelled object α.

The labelled product $\alpha \star \beta$ of two well-labelled objects $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$ is defined by

$$\alpha \star \beta = \{(\alpha', \beta') : \rho_1(\alpha) = \alpha', \rho_2(\beta) = \beta', (\alpha', \beta') \text{ is a well-labelled object}\}, \tag{3.65}$$

which leads to the definition of the labelled product of the labelled classes $\mathcal{A}$ and $\mathcal{B}$

$$\mathcal{A} \star \mathcal{B} = \bigcup_{\substack{\alpha \in \mathcal{A} \\ \beta \in \mathcal{B}}} \alpha \star \beta. \tag{3.66}$$

Consider for example the two graphs $G_1, G_2 \in \mathcal{G}$ from the previously defined labelled class $\mathcal{G}$ depicted in Figure 3.3, whereas $G_1 \star G_2$ is found in Figure 3.4.

Observe that $G_1 \star G_2$ contains exactly $\binom{6}{4} = 15$ elements, as there are exactly $\binom{6}{4}$ ways to choose four new labels among the numbers $\{1, \ldots, 6\}$ for the object G_1 and the object G_2 is relabelled with the remaining two. In general there are $\binom{n}{k}$ possible ways to relabel two objects $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$ with $|\alpha| = k$ and $|\beta| = n - k$ such that $(\rho_1(\alpha), \rho_2(\beta))$ is well-labelled.

The labelled product of two labelled classes $\mathcal{A}$ and $\mathcal{B}$ is easily transfered by the exponential generating functions $\hat{A}(z)$ and $\hat{B}(z)$ of these labelled classes.

Theorem 38. Let $\mathcal{A}$ and $\mathcal{B}$ be two labelled classes then the exponential generating function $\widehat{A \star B}(z)$ of the labelled product of $\mathcal{A}$ and $\mathcal{B}$ is given by $\hat{A}(z)\hat{B}(z)$, where

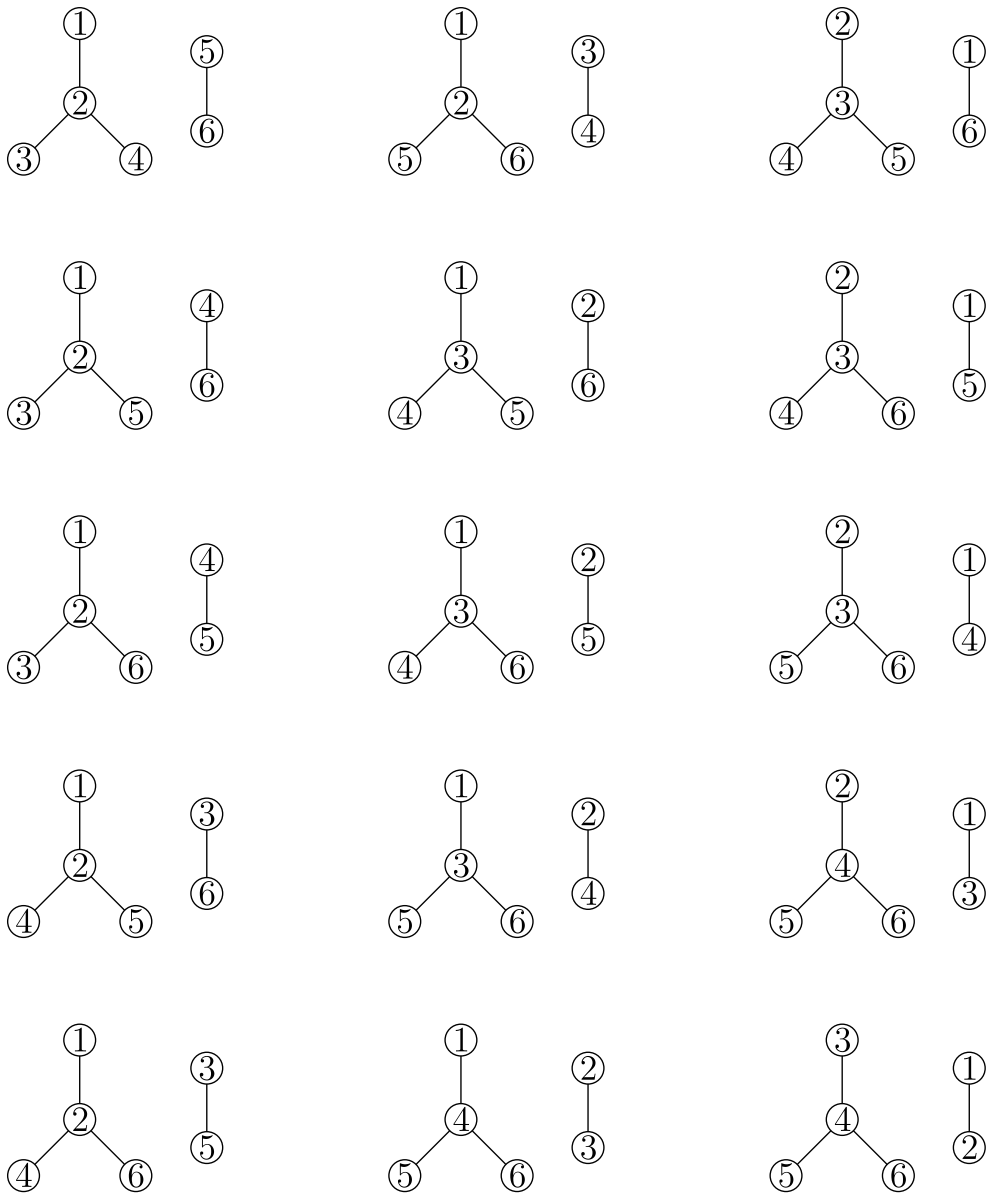

Figure 3.4.: The labelled product of G_1 and G_2

$\hat{A}(z)$ and $\hat{B}(z)$ are denoting the exponential generating functions of $\mathcal{A}$ and $\mathcal{B}$, respectively.

Proof. The proof is readily achieved by considering

$$\begin{aligned}
\widehat{A \star B}(z) &= \sum_{(\alpha,\beta)\in\mathcal{A}\star\mathcal{B}} \frac{z^{|\alpha|+|\beta|}}{(|\alpha|+|\beta|)!} \\
&= \sum_{n\geq 0}\left(\sum_{\substack{|\alpha|+|\beta|=n\\(\alpha,\beta)\in\mathcal{A}\star\mathcal{B}}} 1\right)\frac{z^n}{n!} \\
&= \sum_{n\geq 0}\left(\sum_{\substack{k=0\\ \alpha\in\mathcal{A},\beta\in\mathcal{B}\\ |\alpha|=k,|\beta|=n-k}}^{n} \binom{n}{k}\right)\frac{z^n}{n!} \\
&= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k} a_k b_{n-k}\right)\frac{z^n}{n!} \\
&= \hat{A}(z)\hat{B}(z).
\end{aligned}$$

The third equation utilizes the fact that the cardinality of $\alpha\star\beta$ for $\alpha\in\mathcal{A}$ and $\beta\in\mathcal{B}$ with $|\alpha|=k$ and $|\beta|=n-k$ is $\binom{n}{k}$, as there are $\binom{n}{k}$ possible ways to choose k labels among the numbers $\{1,\ldots,n\}$ to relabel α and the remaining $n-k$ labels to relabel β. □

The operation of the labelled product can be utilized to define a sequence construction, as done in [13]. Let $\mathcal{D}$ be a labelled class and $\mathcal{H}=\mathcal{D}^k$, where $\mathcal{D}^k$ denotes the k-fold labelled product of $\mathcal{D}$. Then $\mathcal{H}$ consists of the well-labelled sequences $h=(\rho(d_1),\ldots,\rho(d_k))\in\mathcal{H}$ of length k with $|h|=|d_1|+\ldots+|d_k|$ and $d_i\in\mathcal{D}$, so that the equation $\mathcal{H}=\mathcal{D}^k$ translates to $\hat{H}(z)=\hat{D}^k(z)$.

It is possible to drop the stipulated length restriction on the sequences by assuming that $\mathcal{H}$ is constructed out of $\mathcal{D}$ by

$$\operatorname{Seq}\{D\} := \sum_{k\geq 0}\mathcal{D}^k, \tag{3.67}$$

where the notation $\mathcal{H}=\operatorname{Seq}\{\mathcal{D}\}$ is taken from [13]. Note that this necessitates $d_0=0$, so that there is no object of size zero in $\mathcal{D}$.

Thus $\mathcal{H}$ denotes the labelled class of all possible sequences, that can be formed from objects in $\mathcal{D}$, where the size of the objects in $\mathcal{H}$ is the sum of the sizes of the objects in $\mathcal{D}$ that the sequence consists of.

The relation between the exponential generating functions of $\mathcal{H}$ and $\mathcal{D}$ can be concluded from the Equation 3.67. Hence

$$\hat{H}(z)=\sum_{k\geq 0}\hat{D}^k(z)=\frac{1}{1-\hat{D}(z)} \tag{3.68}$$

follows.

To exemplify the Seq construction consider the labelled class of all binary words $\mathcal{W}$, where each word $w \in \mathcal{W}$ is a sequence $w = w_1 \dots w_n$ of zeros and ones. Using the above language of sequences the equation

$$\mathcal{W} = \mathrm{Seq}\{\mathcal{Z}_0 + \mathcal{Z}_1\} \tag{3.69}$$

is deduced, where $\mathcal{Z}_0$ and $\mathcal{Z}_1$ are the atomic classes, representing the "zero" and the "one" object.

The exponential generating function is $\hat{W}(z) = \frac{1}{1-2z}$ implying

$$w_n = \left[\frac{z^n}{n!}\right] \frac{1}{1-2z} \tag{3.70}$$

$$= [z^n] \sum_{n \geq 0} 2^n z^n \tag{3.71}$$

$$= 2^n, \tag{3.72}$$

as expected by basic combinatorial reasoning.

Another construction besides the sequence operator arises, if the order of the elements in the sequences is considered to be immaterial, which is reminiscent to a set of objects. In other words two sequences $u, v \in \mathcal{H} = \mathrm{Seq}\{\mathcal{D}\}$ of length k are considered to be equivalent, if a permutation $\pi : \{1, \dots, k\} \to \{1, \dots, k\}$ exists, so that $v = (v_1, \dots, v_k) = (u_{\pi(1)}, \dots, u_{\pi(k)})$ holds.

The derivation of the set operation is quite straightforward if it is observed, that every set $\{\rho(d_1), \dots, \rho(d_k)\}$ emerging from k labelled objects gives rise to exactly $k!$ different sequences $\big(\rho(d_{\pi(1)}), \dots, \rho(d_{\pi(k)})\big)$ under the permutation π defined above. The connection between the labelled class $\mathcal{H}$ of k element subsets of elements from $\mathcal{D}$ is expressed by the equation $\mathcal{H} = \mathrm{Set}_k\{\mathcal{D}\}$ leading to the exponential generating function $\hat{H}(z) = \frac{1}{k!}\hat{D}^k(z)$.

When dropping the restriction of the size of the subset, the arbitrary subset construction

$$\mathrm{Set}\{D\} = \sum_{k \geq 0} \mathrm{Set}_k\{\mathcal{D}\} \tag{3.73}$$

can be considered if it is again assumed that $\mathcal{D}$ does not contain any object of size zero i.e. $d_0 = 0$. The expression $\mathcal{H} = \mathrm{Set}\{\mathcal{D}\}$ leads to

$$\hat{H}(z) = \sum_{k \geq 0} \frac{\hat{D}^k(z)}{k!} = \exp\left\{\hat{D}(z)\right\}, \tag{3.74}$$

which is also called the *exponential formula* as done in [34, Chapter 3].

Sometimes not only the number of objects of a given size n in $\mathcal{H} = \mathrm{Set}\{\mathcal{D}\}$ is of interest, but moreover the number k of parts of $\mathcal{D}$ the elements of $h = \{\rho(d_1), \dots, \rho(d_k)\} \in \mathcal{H}$ are consisting of. In order to account for the number of parts it is necessary to introduce a part counting function ψ on the set $\mathcal{H}$, where for all $h \in \mathcal{H}$ it is

$$\psi(h) = |\{\rho(d_1), \dots, \rho(d_k)\}| \tag{3.75}$$

set.

Following the above argumentation the bivariate generating function $\hat{H}(z,u)$ is defined by

$$\hat{H}(z,u) = \sum_{h\in\mathcal{H}} u^{\psi(h)} \frac{z^{|h|}}{|h|!} \tag{3.76}$$

$$= \sum_{n\geq 0, k\geq 0} h_{n,k} u^k \frac{z^n}{n!} \tag{3.77}$$

$$= \sum_{k\geq 0} u^k \frac{\hat{D}^k(z)}{k!} \tag{3.78}$$

$$= \exp\left\{u\hat{D}(z)\right\}, \tag{3.79}$$

where the coefficient $h_{n,k}$ accounts for the sets $\{\rho(d_1),\ldots,\rho(d_k)\}$ with k elements of size n. Note the equivalence $\hat{H}(z) = \hat{H}(z,1)$. See also [13, chapter 3] for this procedure.

A very good introduction to the wide variety of problems that can be solved using this formula is found in [34] and [13]. It is notable that [34] denotes the function $\hat{D}(z)$ as the deck-enumerator and $\hat{H}(z,u)$ as the hand-enumerator interpreting $\hat{H}(z,u)$ in a more figurative way as a deck of playing cards.

The first example that demonstrates the exponential formula is the counting of connected graphs with n vertices. In this application the hand-enumerator is easily determined if $u = 1$ is set, as

$$\hat{H}(z) = \hat{H}(z,1) = \sum_{n\geq 0} 2^{\binom{n}{2}} \frac{z^n}{n!} = \exp \hat{D}(z) \tag{3.80}$$

holds.

Since there are $2^{\binom{n}{2}}$ graphs having n vertices. Note that $\hat{H}(z)$ can only be viewed as a formal power series, as $\hat{H}(z)$ is not converging for any z except $z = 0$. The deck-numerator $\hat{D}(z)$ is the exponential generating function for the sequence of the number of connected graphs with n vertices and can be determined by applying the $z\frac{d}{dz}\log$ operator on both sides of the Equation 3.80, as done in [34, Chapter 3], leading to

$$z\hat{H}'(z) = z\hat{D}'(z)\hat{H}(z) \tag{3.81}$$

and finally

$$\sum_{n\geq 0} n 2^{\binom{n}{2}} \frac{z^n}{n!} = \left(\sum_{n\geq 0} n d_n \frac{z^n}{n!}\right)\left(\sum_{n\geq 0} 2^{\binom{n}{2}} \frac{z^n}{n!}\right) \tag{3.82}$$

$$= \sum_{n\geq 0}\left(\sum_{k=0}^{n} \binom{n}{k} k d_k 2^{\binom{n-k}{2}} \frac{z^n}{n!}\right) \tag{3.83}$$

resulting in the recurrence relation

$$d_n = 2^{\binom{n}{2}} - \sum_{k=0}^{n-1} d_k \binom{n-1}{k-1} 2^{\binom{n-k}{2}}, \tag{3.84}$$

with the initial value $d_1 = 1$, that is also found in [34, page 86, formula 3.10.2].

The recurrence relation 3.84 can be used to determine the number $c_n := d_n$ of connected graphs with n vertices in a recursive fashion as done in Table 3.4. The numbers c_n are listed in [28, **A001187**].

n	c_n
1	1
2	1
3	4
4	38
5	728
6	26704
7	1866256
8	251548592
9	66296291072
10	34496488594816
11	35641657548953344
12	73354596206766622208
13	301272202649664088951808
14	2471648811030443735290891264
15	40527680937730480234609755344896
16	1328578958335783201008338986845427712
17	87089689052447182841791388989051400978432
18	11416413520434522308788674285713247919244640256
19	2992938411601818037370034280152893935458466172698624

Table 3.4.: Number of connected graphs with n vertices

Even more it is then possible to calculate the number of graphs with n vertices and exactly k components, which are given by the $c_{n,k} := h_{n,k}$, that are listed in Table 3.5 and [28, **A143543**].

Using the Equation 3.42 and the Table 3.5 it is also possible to calculate a table for the numbers $c^e_{n,j}$ and $c^o_{n,j}$ introduced in Section 3.2, as

$$c^e_{n,j} = \frac{1}{2}\Big(c_{n,j} + s(n,j)\Big) \tag{3.85}$$

$$c^o_{n,j} = \frac{1}{2}\Big(c_{n,j} - s(n,j)\Big) \tag{3.86}$$

holds. Tables for the $c^e_{n,j}$ and $c^o_{n,j}$ for small values of n are found in Table 3.6 and Table 3.7, but seemingly not in [28].

With a slight modification the exponential formula can even handle the problem to determine the number of connected graphs with n vertices and i edges, it is only necessary to incorporate a marking variable v in $\hat{H}(z,u)$ to account for the number of edges see also [3, page 123, Excercise 3.55]. Hence

$$\hat{H}(z,u,v) = e^{u\hat{D}(z,v)} \tag{3.87}$$

n	$c_{n,1}$	$c_{n,2}$	$c_{n,3}$	$c_{n,4}$	$c_{n,5}$	$c_{n,6}$	$c_{n,7}$	$c_{n,8}$	$c_{n,9}$
1	1	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0
3	4	3	1	0	0	0	0	0	0
4	38	19	6	1	0	0	0	0	0
5	728	230	55	10	1	0	0	0	0
6	26704	5098	825	125	15	1	0	0	0
7	1866256	207536	20818	2275	245	21	1	0	0
8	251548592	15891372	925036	64673	5320	434	28	1	0
9	66296291072	2343580752	76321756	3102204	169113	11088	714	36	1

Table 3.5.: Number of graphs with n vertices and k components $c_{n,k}$

n	$c^e_{n,1}$	$c^e_{n,2}$	$c^e_{n,3}$	$c^e_{n,4}$	$c^e_{n,5}$	$c^e_{n,6}$	$c^e_{n,7}$	$c^e_{n,8}$	$c^e_{n,9}$
1	1	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0
3	3	0	1	0	0	0	0	0	0
4	16	15	0	1	0	0	0	0	0
5	376	90	45	0	1	0	0	0	0
6	13292	2686	300	105	0	1	0	0	0
7	933488	102886	11221	770	210	0	1	0	0
8	125771776	7952220	455952	35721	1680	378	0	1	0
9	33148165696	1171735584	38219940	1517460	95781	3276	630	0	1

Table 3.6.: Number of graphs with n vertices and k components with an even number of edges $c^e_{n,k}$

becomes the determining equation, which needs to be solved for $\hat{D}(z,v)$. Observe that $\hat{H}(z,v) := \hat{H}(z,1,v)$ is given by

$$\hat{H}(z,v) = \sum_{n\geq 0}(1+v)^{\binom{n}{2}}\frac{z^n}{n!}, \tag{3.88}$$

as there are exactly $\binom{\binom{n}{2}}{i}$ graphs with n vertices and i edges. Applying again the $z\frac{d}{dz}\log$ operator on both sides of Equation 3.87 yields

$$z\frac{\partial H(z,v)}{\partial z} = H(z,v)z\frac{\partial D(z,v)}{\partial z} \tag{3.89}$$

and after some elementary, but tedious transformations, the recurrence relation

$$d_{n,j} = \binom{\binom{n}{2}}{j} - \sum_{k=0}^{n-1}\sum_{i=0}^{j} d_{k,i}\binom{n-1}{k-1}\binom{\binom{n-k}{2}}{j-i} \tag{3.90}$$

is established, where $d_{n,j} := n![z^n v^j]\hat{D}(z,v)$ are the numbers under consideration. The boundary values of this recurrence relation are $d_{0,0} = d_{1,0} = 1$ and $d_{n,0} = 0$

n	$c^o_{n,1}$	$c^o_{n,2}$	$c^o_{n,3}$	$c^o_{n,4}$	$c^o_{n,5}$	$c^o_{n,6}$	$c^o_{n,7}$	$c^o_{n,8}$	$c^o_{n,9}$
1	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0
3	1	3	0	0	0	0	0	0	0
4	22	4	6	0	0	0	0	0	0
5	352	140	10	10	0	0	0	0	0
6	13412	2412	525	20	15	0	0	0	0
7	932768	104650	9597	1505	35	21	0	0	0
8	125776816	7939152	469084	28952	3640	56	28	0	0
9	33148125376	1171845168	38101816	1584744	73332	7812	84	36	0

Table 3.7.: Number of graphs with n vertices and k components with an odd number of edges $c^o_{n,k}$

for $n \geq 2$ and $d_{0,i} = 0$ for $i \geq 1$. The recurrence relation 3.90 is also found in [26, page 613].

Taking a closer look an elementary combinatorial proof of Equation 3.90 is revealed. Firstly, $\binom{\binom{n}{2}}{j}$ counts the number of graphs with n vertices having exactly j edges, but being not necessarily connected, so that the disconnected graphs needed to be subtracted. This is done by partitioning the set of disconnected graphs with n vertices and j edges into disjoint sets. Assume that the partitioning is done by looking at the component of the graph containing the vertex labelled with n. The introduction of the distinguished element n seems to be inevitable to avoid double counting of some configurations, which is reminiscent to the recurrence relation 3.3 of the Bell numbers.

Secondly, there are exactly $\binom{n-1}{k-1}$ ways to choose $k-1$ vertices among the vertices $\{1, \ldots, n-1\}$ and then there are $d_{k,i}$ ways in which this k vertices may be joined by i edges to get a connected subgraph. The remaining part of the graph consists of the remaining $n-k$ vertices having exactly $j-i$ vertices, chosen among the $\binom{n-k}{2}$ possible edges, which gives rise to the factor $\binom{\binom{n-k}{2}}{j-i}$. Summing over all possible number of vertices and edges for the $d_{k,i}$ gives the desired result. Observe that only disconnected graphs are subtracted and that no graph is subtracted twice, by using the distinguished vertex n.

Note that some of the numbers $d_{k,i}$ can be readily obtained, as $d_{n,n-1}$ is simply calculating the number of trees, so that $d_{n,n-1} = n^{n-2}$. Note also that it is always possible to remove less than $n-2$ edges from the complete graph K_n without disconnecting the graph. Hence $d_{n,k} = \binom{n}{k}$ holds, for $k \geq \binom{n}{2} - n + 2$. Quite obviously $d_{n,k}$ vanishes for all $k \leq n-2$ and $k > \binom{n}{2}$. The Table 3.8 and [28, **A062734**] lists the values of all non-vanishing $d_{n,k}$.

Tremendous efforts had been made in the study of the sequences $\{d_{n,n+k}\}_{n\geq 0}$ for $k \geq -1$ using quite sophisticated methods see [18] and [13, page 122ff.].

The second application of the exponential formula yields the exponential generating functions for the Bell numbers and the Stirling numbers of the second kind. In order to use the present framework the Set_Ω operator is defined by

$$\mathrm{Set}_\Omega\{\mathcal{D}\} = \sum_{k\in\Omega} \mathrm{Set}_k\{\mathcal{D}\} \tag{3.91}$$

with $\Omega \subseteq \mathbb{N} \cup \{0\}$, as a generalization of the usual Set operator and the notation $\geq k := \{n \in \mathbb{N} \cup \{0\} : n \geq k\}$ is used to denote the set of all non-negative integers greater or equal k.

Consider the labelled class $\mathcal{B}$ of the nonempty "blocks" defined by

$$\mathcal{B} = \mathrm{Set}_{\geq 1}\{\mathcal{Z}\}\,. \tag{3.92}$$

The labelled class $\mathcal{B} = \{\{1\},\{1,2\},\{1,2,3\},\{1,2,3,4\},\ldots\}$ has the exponential generating function $\hat{B}(z) = \exp(z) - 1$. The elements of $\mathcal{B}$ are serving as building blocks for any set partition, whereas the order of the blocks is by the definition of a set partitions immaterial. Denoting by $\mathcal{P}$ the labelled class of set partitions the relationship

$$\mathcal{P} = \mathrm{Set}\{\mathcal{B}\} = \mathrm{Set}\{\mathrm{Set}_{\geq 1}\{\mathcal{Z}\}\} \tag{3.93}$$

holds see also [13, page 98ff.] for a discussion of this equation. Using this relation the bivariate generating function $\hat{P}(z,u)$ of the class $\mathcal{P}$ is derived, which is given by

$$\hat{P}(z,u) = \sum_{n\geq 0, k\geq 0} S(n,k)u^k \frac{z^n}{n!} = \exp(u(\exp z - 1)). \tag{3.94}$$

Moreover by defining

$$\hat{P}(z) = \sum_{n\geq 0} B(n)\frac{z^n}{n!} = \exp\left(u(\exp z - 1)\right) := \hat{P}(z,1) \tag{3.95}$$

$$\hat{S}^k(z) = \sum_{n\geq 0} S(n,k)\frac{z^n}{n!} = \frac{1}{k!}\left(\exp z - 1\right)^k := \left[u^k\right]\hat{P}(z,u) \tag{3.96}$$

$$B_n(u) = \sum_{k=0}^{n} S(n,k)u^k := n!\,[z^n]\,\hat{P}(z,u) \tag{3.97}$$

the exponential generating functions $\hat{P}(z)$, $\hat{S}^k(z)$ of the Bell numbers and the Stirling numbers of the second kind with a given number k of blocks are found. Fixing the number of elements n defines the *exponential polynomials* $B_n(u)$, which resist to have a more amenable form, see also [34, page 16ff] for information about the exponential polynomials.

Another combinatorial application using a similar approach like the exponential formula, arises in the determination of the exponential generating function, of the unsigned Stirling numbers of the first kind $|s(n,k)|$. Observe that every permutation of a finite set can be decomposed into disjoint cycles and the order of the cycles is immaterial.

In order to derive the exponential generating function of the $|s(n,k)|$ it is necessary to enrich the present framework with another construction. Consider again

$\mathcal{H} = \mathrm{Seq}\{\mathcal{D}\}$ for a labelled class $\mathcal{D}$ and let $u, v \in \mathcal{H}$ be two sequences of length k in $\mathcal{H}$. Assume that u, v are considered to be equivalent if there exists a cyclic permutation $\pi : \{1, \dots, k\} \to \{1, \dots, k\}$, so that $u = (u_1, \dots, u_k) = (v_{\pi(1)}, \dots, v_{\pi(k)})$ holds. A permutation is called cyclic if it has exactly one cylce. Denote by $\mathrm{Cyc}_k\{\mathcal{D}\}$ the labelled class of all k-cyclic arrangements of objects in $\mathcal{D}$. As every cyclic arrangement of length k gives rise to k sequences of length k, the exponential generating functions of $\mathcal{H} = \mathrm{Cyc}\,\{\mathcal{D}\}$ and $\mathcal{D}$ are connected by

$$\hat{H}(z) = \frac{1}{k}\hat{D}^k(z). \tag{3.98}$$

If $\mathcal{H} = \mathrm{Cyc}\{\mathcal{D}\} = \sum_{k\geq 1} \mathrm{Cyc}_k\,\{\mathcal{D}\}$ denotes the labelled class of all cyclic arrangements of elements from $\mathcal{D}$ with arbitrary length, the bivariate generating function $\hat{H}(z, u)$ is

$$\hat{H}(z, u) = \sum_{k\geq 1} \frac{u^k}{k}\hat{D}^k(z) = \ln \frac{1}{1 - u\hat{D}(z)}. \tag{3.99}$$

Note that the cycle construction again necessitates $d_0 = 0$.

The labelled class of all cyclic permutation $\mathcal{C}$ is $\mathcal{C} = \mathrm{Cyc}_{\geq 1}\,\{\mathcal{Z}\}$, so that the labelled class $\mathcal{S}$ of all permutations is given by the equation

$$\mathcal{S} = \mathrm{Set}\,\{\mathcal{C}\} = \mathrm{Set}\left\{\mathrm{Cyc}_{\geq 1}\,\{\mathcal{Z}\}\right\}, \tag{3.100}$$

as the arrangements of the cycles is immaterial. Hence the bivariate generating function $\hat{C}(z, u)$ of the labelled class $\mathcal{C}$ is given by

$$\hat{C}(z, u) = \sum_{n\geq 0, k\geq 0} |s(n, k)| u^k \frac{z^n}{n!} = \exp\left(u \ln \frac{1}{1 - z}\right) = (1 - z)^{-u}. \tag{3.101}$$

Similarly to the results about set partitions the bivariate generating function $\hat{C}(z)$ yields the following generating functions

$$\hat{C}(z) = \sum_{n\geq 0} n! \frac{z^n}{n!} = \frac{1}{1 - z} := \hat{C}(z, 1), \tag{3.102}$$

$$\hat{C}^k(z) = \sum_{n\geq 0} |s(n, k)| \frac{z^n}{n!} = \frac{1}{k!}\left(\ln \frac{1}{1 - z}\right)^k := \left[u^k\right] \hat{C}(z, u), \tag{3.103}$$

$$C_n(u) = \sum_{k=0}^{n} |s(n, k)| u^k = u^{\overline{n}} := n!\,[z^n]\,\hat{C}(z, u). \tag{3.104}$$

The generating functions defined by the equations 3.97 and 3.104 are basically resembling the same idea. Given a labelled class $\mathcal{D}$ and the equation $\mathcal{H} = \mathrm{Set}\,\{\mathcal{D}\}$, then $\Phi_n(u)$ is

$$\Phi_n(u) = \sum_{k\geq 0} h_{n,k} u^k := n!\,[z^n]\,\hat{H}(z, u). \tag{3.105}$$

The $\Phi_n(u)$ are called *polynomials of binomial type*. These polynomials are receiving their names from the identity

$$\Phi_n(u+v) = \sum_{k=0}^{n} \binom{n}{k} \Phi_k(u)\Phi_{n-k}(v), \tag{3.106}$$

holding for all $n \geq 0$ and arbitrary u and v. The identity becomes indeed a triviality if

$$\exp\left\{(u+v)\hat{D}(z)\right\} = \exp\left\{u\hat{D}(z)\right\}\exp\left\{v\hat{D}(z)\right\} \tag{3.107}$$

is considered.

Applying this insight the identity

$$B_n(u+v) = \sum_{k\geq 0} \binom{n}{k} B_k(u)B_{n-k}(v) \tag{3.108}$$

for the exponential polynomials is deduced. For more identities involving polynomials of binomial type see [1], [2, page 99ff.], [24], [23], and [31].

3.6. Bell polynomials

The partial Bell polynomials $B_{n,k}(x_1,\ldots,x_{n-k-1})$ are emerging naturally from the coefficients $h_{n,k}$ of the exponential formula. Assume the exponential generating function $\hat{F}(z)$

$$\hat{F}(z) = \sum_{i\geq 1} x_i \frac{z^i}{i!} \tag{3.109}$$

with the indeterminates x_i, then the *partial Bell polynomials* $B_{n,k}(x_1,\ldots,x_{n-k+1})$ are generated by

$$\sum_{k\geq 0, n\geq 0} B_{n,k}(x_1,\ldots,x_{n-k+1})u^k \frac{z^n}{n!} = \exp\left(u\sum_{i\geq 1} x_i \frac{z^i}{i!}\right). \tag{3.110}$$

The coefficients of the partial Bell polynomials $B_{n,k}(x_1,\ldots,x_{n-k+1})$ can be combinatorial interpreted, as for non-negative integers $k_1,\ldots,k_n$ satisfying $\sum_{i=1}^{k} ik_i = n$ and $\sum_{i=1}^{k} k_i = k$ the coefficient $[x_1^{k_1}\ldots x_n^{k_n}]B_{n,k}(x_1,\ldots,x_{n-k+1})$ is counting the number of set partitions of an n element set with k_i blocks of size i, so that the formula

$$B_{n,k}(x_1,\ldots,x_{n-k+1}) = \sum S(n, 1^{k_1}\ldots n^{k_n})x_1^{k_1}\cdots x_n^{k_n} \tag{3.111}$$

for the partial Bell polynomials holds, where the summation ranges over all n tuples $(k_1,\ldots,k_n)$ satisfying $\sum_{i=1}^{n} ik_i = n$ and $\sum_{i=1}^{n} k_i = k$.

Using this combinatorial viewpoint the recurrence relation for the partial Bell polynomials resembling Theorem 18

$$B_{n+1,k}(x_1,\dots,x_{n+1-k+1}) = \sum_{j=0}^{n-k} \binom{n}{j} x_{j+1} B_{n-j,k-1}(x_1,\dots,x_{n-j-k+1}) \tag{3.112}$$

is given. Utilizing the recurrence relation results in a list of the first partial Bell polynomials printed in Table 3.9. A more exhaustive list is found in [8] and [6].

The *complete Bell polynomials* $B_n(x_1,\dots,x_n)$ are defined by

$$B_n(x_1,\dots,x_n) = \sum_{k=1}^{n} B_{n,k}(x_1,\dots,x_{n-k+1}). \tag{3.113}$$

The following four identities are also readily verified

$$S(n,k) = B_{n,k}(1,\dots,1) \tag{3.114}$$
$$|s(n,k)| = B_{n,k}(0!,\dots,(n-1)!) \tag{3.115}$$
$$B_n(z) = B_n(z,\dots,z) \tag{3.116}$$
$$B(n) = B_n(1,\dots,1). \tag{3.117}$$

3.7. Iterates of formal power series

An interesting application involving the partial Bell polynomials is found in [8], where the structure of α-fold mappings is analyzed. This work is presented here in a brief form and is used to derive a closed formula for the Möbius function in the partition lattice in Chapter 4. When considering formal power series $f(z) \in \mathbb{C}[[z]]$ in this section it is always assumed that they are presented in the exponential form $f(z) = \sum_{k\geq 0} f_k \frac{z^k}{k!}$.

Let $f \in \mathbb{C}[[z]]$ be a formal power series with $f_0 = 0$, then for every integer $\alpha \geq 1$ the α-fold mapping $f^{\langle\alpha\rangle}(z)$ is recursively defined by

$$f^{\langle\alpha+1\rangle}(z) = f\left(f^{\langle\alpha\rangle}(z)\right) \tag{3.118}$$

for all $\alpha \geq 2$ with the initial condition $f^{\langle 1\rangle}(z) = f(z)$. To every formal power series $f(z)$ the infinite lower triangle *iteration matrix* is associated

$$\mathbf{B}(f) = \begin{pmatrix} B_{1,1} & 0 & 0 & \dots \\ B_{2,1} & B_{2,2} & 0 & \dots \\ B_{3,1} & B_{3,2} & B_{3,3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3.119}$$

where the $B_{n,k} := B_{n,k}(f_1,\dots,f_{n-k+1})$ are the partial Bell polynomials in f_i. The following theorem is an immediate consequence of this definition and is taken from [8].

Theorem 39. Let $f, g, h \in \mathbb{C}[[z]]$ be formal power series, with $g_0 = 0$ and let $h(z) = (f \circ g)(z)$, then the iteration matrices of f, g and h are connected by the matrix product

$$\mathbf{B}(h) = \mathbf{B}(g)\mathbf{B}(f). \tag{3.120}$$

Proof. For each $k \geq 1$

$$\begin{aligned}
\sum_{n\geq k} B_{n,k}(h_1, \dots, h_{n-k+1})\frac{z^n}{n!} &= \frac{1}{k!}h(z)^k \\
&= \frac{1}{k!}\left(f(g(z))\right)^k \\
&= \sum_{l\geq k} B_{l,k}(f_1, \dots, f_{l-k+1})\frac{g(z)^l}{l!} \\
&= \sum_{n\geq l\geq k} B_{n,l}(g_1, \dots, g_{n-l+1})B_{l,k}(f_1, \dots, f_{l-k+1})\frac{z^n}{n!}
\end{aligned}$$

holds. Comparing the coefficients of $\frac{z^n}{n!}$ proves the claim. □

When considering the first column of the matrix equation the following corollary is deduced, that is also known as the formula of Faà di Bruno see also [8].

Corollary 40. Let $f, g, h \in \mathbb{C}[[z]]$ with $g_0 = 0$ and $h(z) = (f \circ g)(z)$ then the coefficients h_n of h satisfy

$$h_n = \sum_{k\geq 0} f_k B_{n,k}(g_1, \dots, g_{n-k+1}). \tag{3.121}$$

The theorem could be used to derive an explicit representation of the *inverse Bell polynomials* $L_{n,k}$, which are known as the *logarithmic Bell polynomials* and the *potential polynomials* $P_n^{(\lambda)}$, see [6] and [8] for exhaustive material about these polynomials.

Let $g \in \mathbb{C}[[z]]$ be a formal power series with $g_0 = 0$, then the $L_{n,k}$ are defined by

$$h(z) = \sum_{n\geq 0, k\geq 0} L_{n,k}(g_1, \dots, g_{n-k+1})u^k\frac{z^n}{n!} = \ln\left(1 + ug(z)\right). \tag{3.122}$$

Observe that $f(z) = \ln(1 + z)$ with $f(z) = \sum_{n\geq 1}(-1)^{n-1}(n-1)!\frac{z^n}{n!}$ implies

$$L_{n,k}(g_1, \dots, g_{n-k+1}) = (-1)^{k-1}(k-1)!B_{n,k}(g_1, \dots, g_{n-k+1}). \tag{3.123}$$

The defining equation for the Potential polynomials $P_n^{(\lambda)}$ is given by $f(z) = (1+z)^\lambda$

$$1 + \sum_{n\geq 1} P_n^{(\lambda)}\frac{z^n}{n!} = \left(1 + g(z)\right)^\lambda. \tag{3.124}$$

where $\lambda \in \mathbb{C}$ and $g_0 = 0$ is assumed. Utilizing the binomial theorem yields

$$1 + \sum_{n\geq 1} P_n^{(\lambda)} \frac{z^n}{n!} = \sum_{k\geq 0} \binom{\lambda}{k} g^k(z), \tag{3.125}$$

so that the $P_n^{(\lambda)}$ could be expressed by

$$P_n^{(\lambda)}(g_1, \ldots, g_n) = \sum_{k=1}^{n} \binom{\lambda}{k} k! B_{n,k}(g_1, \ldots, g_{n-k+1}) \tag{3.126}$$

or if $r = \lambda$ ([31] formula 1.4.) is a positive integer by

$$P_n^{(r)}(g_1, \ldots, g_n) = \binom{n+r}{r}^{-1} B_{n+r,r}(1, 2g_1, 3g_2, \ldots, (n+1)g_n). \tag{3.127}$$

Another interesting application is derived when considering α-fold mappings, as the Theorem 39 then states $\mathbf{B}(f^{\langle\alpha\rangle}) = [\mathbf{B}(f)]^{\alpha}$ for all natural numbers α.

Assuming that $f_0 = 0$ and $f_1 = 1$ the matrix $\mathbf{B}(f) = \mathbf{I} + \mathbf{C}(f)$ could be written as the sum of the infinite identity matrix $\mathbf{I}$ and the matrix $\mathbf{C}$, which allows to extend α-fold mappings to arbitrary values for $\alpha \in \mathbb{C}$ by

$$[\mathbf{B}(f)]^{\alpha} = [\mathbf{I} + \mathbf{C}(f)]^{\alpha} = \sum_{j\geq 0} \binom{\alpha}{j} \mathbf{C}(f)^j, \tag{3.128}$$

as exemplified in [8]. If furthermore $\left\{[\mathbf{B}(f)]^{\alpha}\right\}_{n,k}$ denotes the element of $\mathbf{B}(f)$, that is found in the n-th row and and k-th column,

$$\left\{[\mathbf{B}(f)]^{\alpha}\right\}_{n,k} = \sum_{j=0}^{n-k} \binom{\alpha}{j} \left\{\mathbf{C}(f)^j\right\}_{n,k} \tag{3.129}$$

holds. Considering the special case $k = 1$ and defining

$$c_{n,j}(f_1, \ldots, f_{n-j+1}) = \left\{\mathbf{C}(f)^j\right\}_{n,1} \tag{3.130}$$

the power series representation of the α-fold mapping $f^{<\alpha>}(z)$ of $f(z)$ for arbitrary fractionary values of α with $f_1 = 1$ and $f_0 = 0$ is given by

$$f^{\langle\alpha\rangle}(z) = \sum_{n\geq 0} \left(\sum_{j=0}^{n-1} \binom{\alpha}{j} c_{n,j}(f_1, \ldots, f_{n-j+1}) \right) \frac{z^n}{n!} \tag{3.131}$$

and the representation of the compositional inverse $f^{\langle -1\rangle}$ of f by setting $\alpha = -1$, is

$$f^{<-1>}(z) = \sum_{n\geq 0} \left(\sum_{j=0}^{n-1} (-1)^j c_{n,j}(f_1, \ldots, f_{n-j+1}) \right) \frac{z^n}{n!}. \tag{3.132}$$

The first values for the polynomials $c_{n,k}$ are found in Table 3.10 taken from [8].

$$c_{1,0} = 1 \quad c_{2,1} = f_2 \quad c_{3,1} = f_3 \quad c_{3,2} = 3f_2^2 \quad c_{4,1} = f_4 c_{4,2} = 3f_2^3 + 10f_2f_3 c_{4,3} = 18f_2^3$$
$$c_{5,1} = f_5 \quad c_{5,2} = 25f_2^2f_3 + 15f_2f_4 + 10f_3^2 \quad c_{5,3} = 75f_2^4 + 130f_2^2f_3 \quad c_{5,4} = 180f_2^4$$
$$c_{6,1} = f_6 \quad c_{6,2} = 70f_2f_3^2 + 60f_2^2f_4 + 21f_2f_5 + 15f_3f_2^3 + 35f_3f_4$$
$$c_{6,3} = 180f_2^5 + 1065f_3f_2^3 + 270f_2^2f_4 + 350f_2f_3^2 \quad c_{6,4} = 1935f_2^5 + 2310f_3f_2^3 \quad c_{6,5} = 2700f_2^5$$
$$c_{7,1} = f_7 \quad c_{7,2} = 350f_2f_3f_4 + 126f_2^2f_5 + 28f_2f_6 + 105f_2^2f_3^2 + 70f_3^3 + 56f_3f_5 + 105f_4f_2^3 + 35f_4^2$$
$$c_{7,3} = 4935f_2^4f_3 + 5705f_2^2f_3^2 + 3255f_4f_2^3 + 504f_2^2f_5 + 315f_2^6 + 1610f_2f_3f_4 + 350f_3^3$$
$$c_{7,4} = 13545f_2^6 + 42420f_2^4f_3 + 6300f_4f_2^3 + 11900f_2^2f_3^2$$
$$c_{7,5} = 59535f_2^6 + 54810f_2^4f_3 \quad c_{7,6} = 56700f_2^6$$

Table 3.10.: $c_{n,k}(f_1, \ldots, f_{n-k+1})$ polynomials

It is already mentioned here that the polynomials $c_{n,k}(f_1, \ldots, f_{n-k+1})$ are generalizations of the numbers $e_{n,k}$. The $e_{n,k}$ are defined in Section 4.1 and are counting the number of chains without repetitions in the partition lattice.

j	$d_{9,j}$	j	$d_{9,j}$	j	$d_{9,j}$	j	$d_{9,j}$
8	4782969	16	7032842901	24	1251493425	32	58905
9	33779340	17	8403710364	25	600775812	33	7140
10	136368414	18	8956859646	26	254183454	34	630
11	405918324	19	8535294180	27	94143028	35	36
12	974679363	20	7279892361	28	30260331	36	1
13	1969994376	21	5557245480	29	8347680		
14	3431889000	22	3792906504	30	1947792		
15	5228627544	23	2309905080	31	376992		
j	$d_{8,j}$	j	$d_{8,j}$	j	$d_{8,j}$	j	$d_{8,j}$
7	262144	13	35804384	19	6905220	25	3276
8	1436568	14	39183840	20	3107937	26	378
9	4483360	15	37007656	21	1184032	27	28
10	10230360	16	30258935	22	376740	28	1
11	18602136	17	21426300	23	98280		
12	28044072	18	13112470	24	20475		
j	$d_{7,j}$	j	$d_{7,j}$	j	$d_{7,j}$	j	$d_{7,j}$
6	16807	10	331506	14	116175	18	1330
7	68295	11	343140	15	54257	19	210
8	156555	12	290745	16	20349	20	21
9	258125	13	202755	17	5985	21	1
j	$d_{6,j}$	j	$d_{6,j}$	j	$d_{6,j}$	j	$d_{6,j}$
5	1296	8	6165	11	1365	14	15
6	3660	9	4945	12	455	15	1
7	5700	10	2997	13	105		
j	$d_{5,j}$	j	$d_{5,j}$	j	$d_{5,j}$	j	$d_{5,j}$
4	125	6	205	8	45	10	1
5	222	7	120	9	10		
j	$d_{4,j}$	j	$d_{4,j}$	j	$d_{4,j}$	j	$d_{4,j}$
3	16	4	15	5	6	6	1
j	$d_{3,j}$	j	$d_{3,j}$	j	$d_{2,j}$	j	$d_{1,j}$
2	3	3	1	1	1	0	1

Table 3.8.: Number of connected graphs with n vertices and j edges

$B_{0,0} = 1$

$B_{1,1} = x_1 \quad B_{2,1} = x_2 \quad B_{2,2} = x_1^2$

$B_{3,1} = x_3 \quad B_{3,2} = 3x_1x_2 \quad B_{3,3} = x_1^3$

$B_{4,1} = x_4 \quad B_{4,2} = 3x_2^2 + 4x_1x_3 \quad B_{4,3} = 6x_1^2x_2 \quad B_{4,4} = x_1^4$

$B_{5,1} = x_5 \quad B_{5,2} = 10x_2x_3 + 5x_1x_4 \quad B_{5,3} = 15x_1x_2^2 + 10x_1^2x_3 \quad B_{5,4} = 10x_1^3x_2 \quad B_{5,5} = x_1^5$

$B_{6,1} = x_6 \quad B_{6,2} = 10x_3^2 + 15x_2x_4 + 6x_1x_5 \quad B_{6,3} = 15x_2^3 + 60x_1x_2x_3 + 15x_1^2x_4 \quad B_{6,4} = 45x_1^2x_2^2 + 20x_1^3x_3 \quad B_{6,5} = 15x_1^4x_2 \quad B_{6,6} = x_1^6$

$B_{7,1} = x_7 \quad B_{7,2} = 35x_3x_4 + 21x_2x_5 + 7x_1x_6 \quad B_{7,3} = 105x_2^2x_3 + 70x_1x_3^2 + 105x_1x_2x_4 + 21x_1^2x_5$

$B_{7,4} = 105x_1x_2^3 + 210x_1^2x_2x_3 + 35x_1^3x_4 \quad B_{7,5} = 105x_1^3x_2^2 + 35x_1^4x_3 \quad B_{7,6} = 21x_1^5x_2 \quad B_{7,7} = x_1^7$

$B_{8,1} = x_8 \quad B_{8,2} = 35x_4^2 + 56x_3x_5 + 28x_2x_6 + 8x_1x_7 \quad B_{8,3} = 280x_2x_3^2 + 210x_2^2x_4 + 280x_1x_3x_4 + 168x_1x_2x_5 + 28x_1^2x_6$

$B_{8,4} = 105x_2^4 + 840x_1x_2^2x_3 + 280x_1^2x_3^2 + 420x_1^2x_2x_4 + 56x_1^3x_5 \quad B_{8,5} = 420x_1^2x_2^3 + 560x_1^3x_2x_3 + 70x_1^4x_4$

$B_{8,6} = 210x_1^4x_2^2 + 56x_1^5x_3 \quad B_{8,7} = 28x_1^6x_2 \quad B_{8,8} = x_1^8$

Table 3.9.: Partial Bell polynomials

4. The partition lattice

The set of all set partitions $\Pi(X)$ of a finite set X can be partially ordered. Let $\pi, \sigma \in \Pi(X)$ and set $\pi \leq \sigma$ if for every block $B_i \in \pi$ there is a block $C_j \in \sigma$, such that $B_i \subseteq C_j$ holds. Then $\leq$ is a reflexive, transitive and antisymmetric relation on $\Pi(X)$ and $(\Pi(X), \leq)$ is a partially ordered set.

Moreover the abbreviated form $i_1^1 \dots i_{s_1}^1 | \dots | i_1^k \dots i_{s_k}^k$ is used to denote the elements $\pi \in \Pi_n$, where the i_j^k are the n distinct elements from $\{1, \dots, n\}$, and all the elements i_j^k for a fixed k and $1 \leq j \leq s_k$ are in the same block among the k blocks in π.

The order relation $\leq$ can also be restated in terms of the equivalence relation of Section 3.1. Let $\pi, \sigma \in \Pi_n$ then $\pi \leq \sigma$ iff for all $x, y \in \{1, \dots, n\}$ with $x \sim_\pi y$ always $x \sim_\sigma y$ holds.

The minimum $\hat{0}$ of Π_n is the set partition with n singleton blocks and the maximum $\hat{1}$ of Π_n is the set partition that possesses only one block.

It can be shown that Π_n is not only a poset, but moreover a lattice. The proof is taken from [7, page 3].

Theorem 41. The poset $(\Pi_n, \leq)$ is a lattice.

Proof. Firstly, it is shown that Π_n is a finite meet-semilattice. Let $\pi, \sigma \in \Pi_n$ and $\gamma \in \Pi_n$, so that

$$B_\gamma(x) := B_\pi(x) \cap B_\sigma(x)$$

holds for all $x \in \{1, \dots, n\}$. Note that γ is a lower bound for π and σ.

Secondly, it is proven that γ is the greatest lower bound of π and σ. Let $\alpha \in \Pi_n$ be another lower bound for π and σ, then $B_\alpha(x) \subseteq B_\pi(x), B_\sigma(x)$ and thus

$$B_\alpha(x) \subseteq B_\pi(x) \cap B_\sigma(x) = B_\gamma(x) \tag{4.1}$$

for all $x \in \{1, \dots, n\}$ implying $\alpha \leq \gamma$, so that $\gamma = \pi \wedge \sigma$ follows.

As Π_n is a finite meet-semilattice containing a maximal element $\hat{1}$ the Theorem 7 proves that Π_n is a lattice. □

Utilizing the Theorem 41 a simple copy and paste argument characterizes the infimum in terms of the equivalence relation. Moreover the theorem could be used to construct an efficient algorithm to compute the infimum, as done in [21].

Theorem 42. Let $\pi, \sigma \in \Pi(X)$ and $x, y \in X$ then the following two statements are equivalent

1. $x \sim_\pi y$ and $x \sim_\sigma y$,

2. $x \sim_{\pi \wedge \sigma} y$.

Proof. Observe that $x, y \in B_\pi(x), B_\sigma(x)$ implies $x, y \in B_\pi(x) \cap B_\sigma(x)$ and by utilizing the proof of Theorem 41 it follows that $x, y \in B_{\pi\wedge\sigma}(x)$, so that x and y are in the same block in $\pi \wedge \sigma$ or $x \sim_{\pi\wedge\sigma} y$.

Conversely $x, y \in B_{\pi\wedge\sigma}(x)$ implies $x, y \in B_\pi(x)$ and $x, y \in B_\sigma(x)$ and the claim is proven. □

The Theorem 41 establishes the existence of a supremum $\pi \vee \sigma$ for all $\pi, \sigma \in \Pi_n$, but it does not characterize its construction. The notion of π, σ-overlapping sequences allows a better understanding of the supremum.

Definition 43. Let $\pi, \sigma \in \Pi_n$ and let $B_1, \ldots, B_k$ be a sequence of distinct blocks from π and σ satisfying $B_i \cap B_{i+1} \neq \emptyset$ for all i with $1 \leq i \leq k-1$ then $B_1, \ldots, B_k$ is called a π, σ-overlapping sequence. Denote by $\mathcal{S}_{\pi,\sigma}(x)$ the set of all π, σ-overlapping sequences that are containing at least one block B_i with $x \in B_i$ for a fixed $x \in \{1, \ldots, n\}$.

The connection between the upper bounds $\alpha \in \Pi_n$ of π and σ with $\pi, \sigma \in \Pi_n$ and the π, σ-overlapping sequences can be stated in a theorem.

Theorem 44. Let $\pi, \sigma \in \Pi_n$ then the following two statements are equivalent.

1. $\alpha \in \Pi_n$ is an upper bound for π and σ.
2. For all $x \in \{1, \ldots, n\}$ the block $A(x) \in \alpha \in \Pi_n$ satisfies

$$A(x) \supseteq \bigcup_{(B_1,\ldots,B_k)\in\mathcal{S}_{\pi,\sigma}(x)} \bigcup_{i=1}^{k} B_i. \tag{4.2}$$

Proof. $1 \Rightarrow 2$: Let α be an upper bound for π and σ then for every two blocks $B \in \pi$ and $C \in \sigma$ satisfying $B \cap C \neq \emptyset$ there must be a block $A \in \alpha$ with $A \supseteq B \cup C$. By induction one can verify that this condition can be extended to π, σ-overlapping sequences $B_1, \ldots, B_k$. As every π, σ-overlapping sequence must satisfy this condition the sufficiency is shown

$2 \Rightarrow 1$: The second part of the proof is done by contraposition. Hence assume that α is not an upper bound for π and σ. Without loss of generality there must be a block $B(x) \in \pi$, so that $A(x) \not\supseteq B(x)$ with $A(x) \in \alpha$ for some $x \in \{1, \ldots, n\}$. As $B(x)$ itself is an π, σ-overlapping sequence the necessity is established. □

With the help of Theorem 44 the characterization of the supremum is readily achieved.

Theorem 45. Let $\pi, \sigma \in \Pi_n$ then the following two statements are equivalent

1. $\gamma \in \Pi_n$ is the supremum of π and σ.
2. For all $x \in \{1, \ldots, n\}$ the block $C(x) \in \gamma \in \Pi_n$ satisfies

$$C(x) = \bigcup_{(B_1,\ldots,B_k)\in\mathcal{S}_{\pi,\sigma}(x)} \bigcup_{i=1}^{k} B_i \tag{4.3}$$

.

Proof. According to Theorem 44 it is readily achieved, that γ must be an upper bound of π and σ and moreover that every upper bound α of π and σ satisfies $\gamma \leq \alpha$, as for all $x \in \{1, \ldots, n\}$ the condition $A(x) \supseteq C(x)$ is satisfied, if $A(x) \in \alpha$ and $C(x) \in \gamma$ are denoting the blocks containing x in α and γ, respectively. □

In the same vein another copy and paste argument states the supremum in terms of the equivalence relation by utilizing the Theorem 45.

Theorem 46. Let $\pi, \sigma \in \Pi_n$ then $x \sim_{\pi \vee \sigma} y$ iff there are $x = z_1, \ldots, z_k = y$ so that $z_i \sim_\pi z_{i+1}$ or $z_i \sim_\sigma z_{i+1}$ holds for all i with $1 \leq i \leq k-1$.

Proof. Utilizing the Theorem 45 it is concluded, that there must be a block $y \in C(x)$ and furthermore a sequence $B_1, \ldots, B_k \in \mathcal{S}_{\pi,\sigma}(x)$ with $x \in B_1$ and $y \in B_k$ and $z_i \in B_i \cap B_{i+1} \neq \emptyset$ for all i with $1 \leq i \leq k-1$, so that $z_i \sim_\sigma z_{i+1}$ or $z_i \sim_\pi z_{i+1}$ follows. The opposite direction could be concluded likewise which proves the claim. □

The Hasse diagramm of the lattice Π_5 is depicted in figure 4.1. It is shown that the partition lattice is semimodular using a theorem taken from [2, page 48].

Theorem 47. The partition lattice Π_n is semimodular.

Proof. Let $\pi, \sigma \in \Pi_n$ and $\pi \wedge \sigma \lessdot \pi, \sigma$ with $\pi \wedge \sigma = \{B_1, B_2, \ldots, B_k\}$ so that without loss of generality it follows $\pi = \{B_1 \cup B_2, B_3, \ldots, B_k\}$ and either $\sigma = \{B_1, B_2 \cup B_3, \ldots, B_k\}$ or $\sigma = \{B_1, B_2, B_3 \cup B_4, \ldots, B_k\}$. The first case yields $\pi \vee \sigma = \{B_1 \cup B_2 \cup B_3, B_4, \ldots, B_k\}$ and the second one $\pi \vee \sigma = \{B_1 \cup B_2, B_3 \cup B_4, \ldots B_k\}$. Both cases fulfill the conditions of semimodularity $\pi, \sigma \lessdot \pi \vee \sigma$. □

As Π_n is semimodular and has a minimum $\hat{0}$ there exists a rank function $r(\pi)$ for all $\pi \in \Pi_n$ given by $r(\pi) = n - |\pi|$.

Theorem 48. The lattice Π_n is a point lattice.

Proof. Let $\pi \in \Pi_n$ then take the supremum $\pi = \sup \pi_{x,y}$ for all $x, y \in \{1, \ldots, n\}$ with $x \sim_\pi y$. □

Corollary 49. The partition lattice Π_n is a finite, semimodular, point lattice and therefore geometric.

As Π_n is a semimodular lattice it satisfies the semimodular inequaltiy. But for a special lattice it always makes sense to think about a direct proof that may reveal more about the structure of Π_n.

The first proof presented here resembles basically the previous theorem about the characterization of semimodularity and is taken from [2] and [21].

The second proof uses a graph theoretic approach and is found by the author himself and does not seem to appear across the literature studied for this thesis.

The semimodular inequality in the partition lattice assumes the following concise form.

Theorem 50. Let $\pi, \sigma \in \Pi_n$ then

$$|\pi| + |\sigma| \leq |\pi \vee \sigma| + |\pi \wedge \sigma| \tag{4.4}$$

holds.

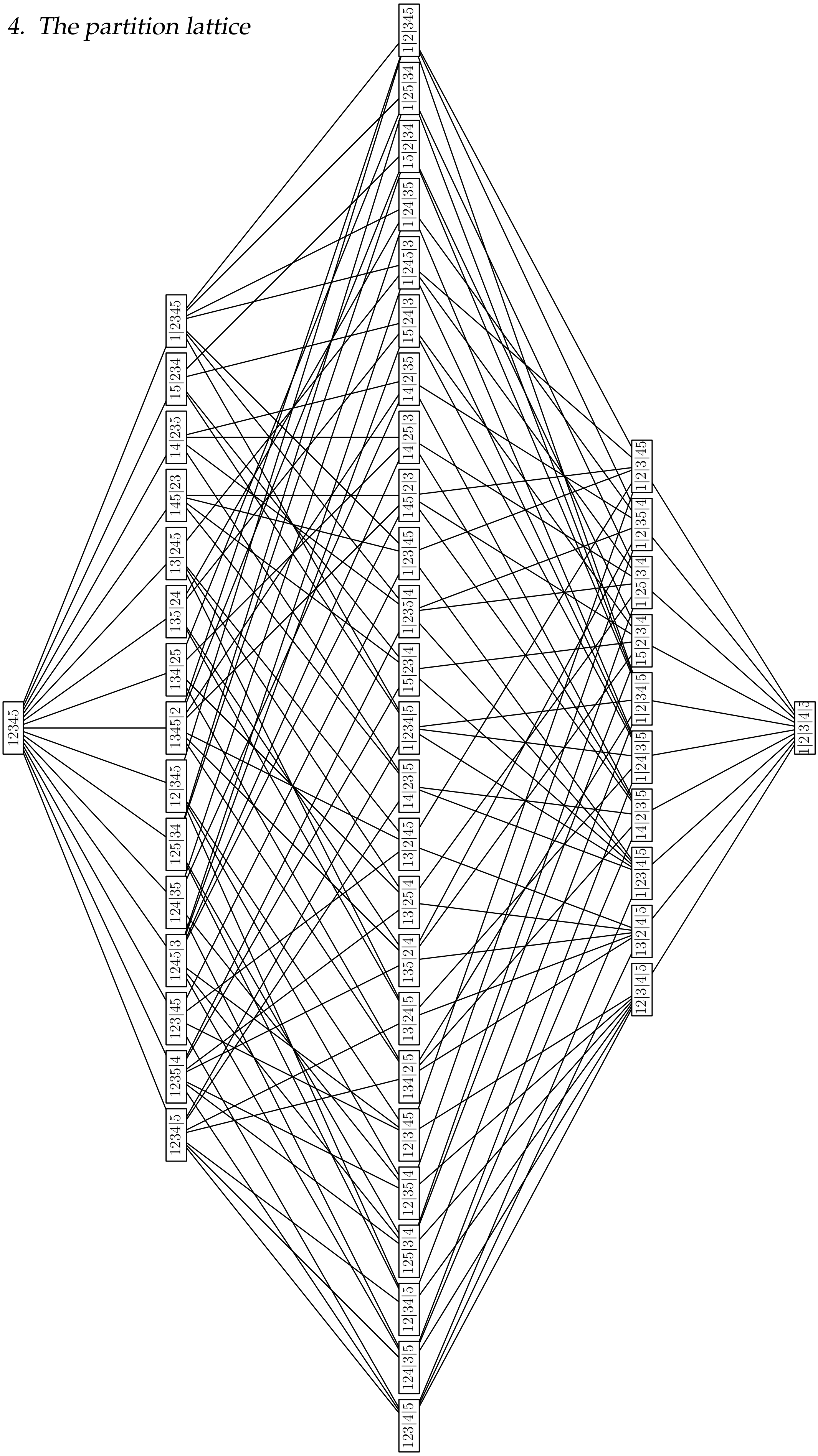

Figure 4.1.: Partition lattice Π_5

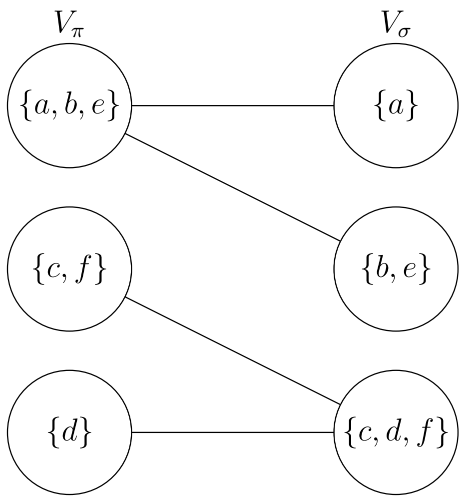

Figure 4.2.: An example for the construction of the bipartite graph $G_{\pi,\sigma}$

Proof 1. π emerges from $\pi \wedge \sigma$ after successively merging blocks in $\pi \wedge \sigma$. Every merge operation is either corresponding to exactly one or no merge operation that are leading σ to $\pi \vee \sigma$. Therefore $|\pi \wedge \sigma| - |\pi| \geq |\sigma| - |\sigma \vee \pi|$, as the left-hand side counts the number of merge operations from $\pi \wedge \sigma$ to π and the right-hand side counts the number of merge operations leading from σ to $\pi \vee \sigma$. □

The idea of the second proof basically uses a transformation of the set partitions π, σ to a bipartite graph $G_{\pi,\sigma}$.

Proof 2. Let $\pi, \sigma \in \Pi_n$ with $|\pi| = r$ and $|\sigma| = s$. Consider the bipartite graph $G_{\pi,\sigma}(V_\pi \cup V_\sigma, E)$, where V_π is identified with the r blocks $B_i \in \pi$ and likewise V_σ with the s blocks $C_j \in \sigma$. The graph contains the edge $\{B_i, C_j\} \in E$ iff $B_i \cap C_j \neq \emptyset$ for all $1 \leq i \leq r$ and $1 \leq j \leq s$. A pictural representation of a special graph $G_{\pi,\sigma}$ with $\pi = abe|cf|d$ and $\sigma = a|be|cdf$ is found in Figure 4.2. The key to the proof are the following bijections, which are readily verified by earlier theorems.

1. The number of edges $|E_{\pi,\sigma}|$ in $G_{\pi,\sigma}$ equals $|\pi \wedge \sigma|$ according to Theorem 42.

2. The number of components $c(G_{\pi,\sigma})$ of $G_{\pi,\sigma}$ equals $|\pi \vee \sigma|$ according to Theorem 46.

Proceed with the theorem that a graph $G(V, E)$ with $c(G)$ components must have at least $|V| - c(G)$ edges thus $|E| \geq |V| - c(g)$. Applying this inequality together with the previous bijections

$$\begin{aligned} c(G) + |E_{\pi,\sigma}| &\geq |V| \\ c(G) + |E_{\pi,\sigma}| &\geq |V_\pi| + |V_\sigma| \\ |\pi \wedge \sigma| + |\pi \vee \sigma| &\geq |\pi| + |\sigma| \end{aligned}$$

yields the semimodular inequality. □

Recall the definitions made in Section 3.2. Due to Definition 27 the graph $G(V,E)$ induces a set partition $\{V_1,\dots,V_k\} = \pi(G) \in \Pi(V)$, where the blocks V_i are the component sets of G. Moreover the edges $\{i,j\} \in E$ of G can be identified with the atomic elements $\pi_{i,j} \in \Pi(V)$ and due to Theorem 46 the equivalence $\pi(G) = \sup_{\{i,j\}\in E} \pi_{i,j}$ is satisfied.

Even though this connection between set partitions and graphs seems to be trivial it is convenient to switch between these two equivalent descriptions.

To illustrate the connection consider the graph depicted in Figure 3.1 and its induced set partition $\pi(G) = \{\{1,2,3,4,5,7,8,9\},\{6,10,11\}\}$ with two blocks that is the supremum of the twelve atomic elements

$$\begin{aligned}\pi(G) = \pi_{1,3} \vee \pi_{1,4} \vee \pi_{2,3} \vee \pi_{2,5} \vee \pi_{2,8} \vee \pi_{3,9} \vee \\ \pi_{4,9} \vee \pi_{5,7} \vee \pi_{5,9} \vee \pi_{6,10} \vee \pi_{6,11} \vee \pi_{7,8}\end{aligned} \tag{4.5}$$

Finally it is mentioned that there is a one-to-one correspondence between the number of graphs with n vertices and j edges with exactly k components and the number of set partitions of an n element set with k blocks that are are a supremum of j atomic elements.

4.1. Chains in the partition lattice

Using the results of Section 2.3 it is possible to calculate the number of chains in Π_n and to derive recurrence relations for them.

Let $\pi,\sigma \in \Pi_n$ with $\pi \le \sigma$ then $z_k(\pi,\sigma)$ denotes the number of chains with repetitions between π and σ of length k. In order to calculate these numbers it is necessary to examine the structure of intervals in Π_n first.

The next theorem characterizes intervals in Π_n as a product order of smaller partition lattices see [32, page 182].

Theorem 51. Let $\pi,\sigma \in \Pi_n$ with $\pi \le \sigma$, then the interval $[\pi,\sigma]$ is isomorphic to a product order with $|\sigma|$ factors

$$[\pi,\sigma] \simeq \coprod_{C_i\in\sigma} \Pi_{p_i} \tag{4.6}$$

with the numbers p_i defined by

$$p_i = |\{B_j \in \pi : B_j \subseteq C_i \in \sigma\}| \quad \text{for all } C_i \in \sigma. \tag{4.7}$$

Proof. Simply take into account that every interval $[\pi,\sigma]$ in Π_n is isomorphic to a product order of filters $\mathfrak{F}(\pi)$ as been exemplified in Figure 4.3.

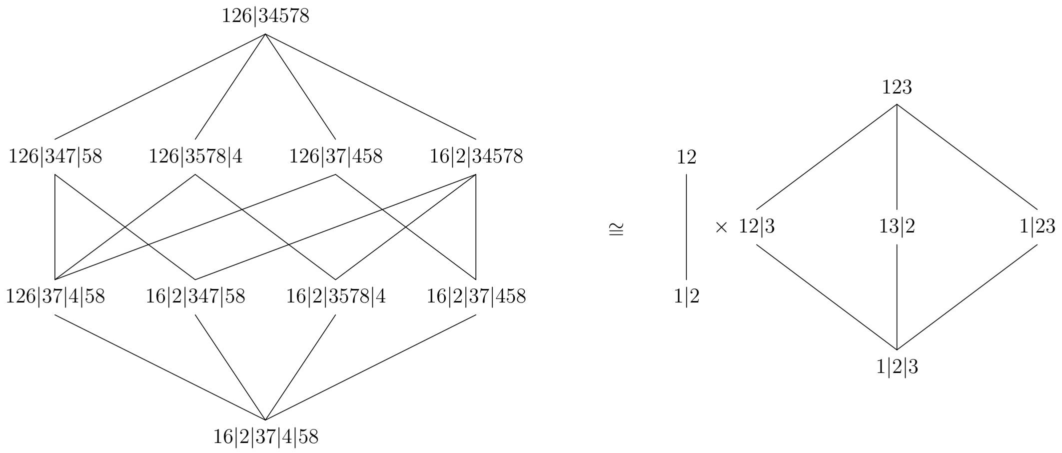

Figure 4.3.: Pictorial illustration of the isomorphism

While taking a look at Figure 4.3 realize that for every filter $\mathfrak{F}(\pi)$ with $\pi \in \Pi_n$ it is $\mathfrak{F}(\pi) \simeq \Pi_{|\pi|}$ by interpreting the blocks of π as the singletons in $\Pi_{|\pi|}$. □

Due to Theorem 51 the attention can be restricted to intervals of the form $[\hat{0}, \hat{1}]$ in Π_n, as arbitrary chains can be constructed from this basic building blocks. The number of chains with repetitions in Π_n between $\hat{0}$ and $\hat{1}$ with length k is denoted by $z_{n,k}$.

The connection between the numbers $z_k(\pi, \sigma)$ and $z_{n,k}$ is then given by

$$z_k(\pi, \sigma) = \sum_{\substack{k_1+\ldots+k_m=k \\ k_i \geq 0}} \binom{k}{k_1, \ldots, k_m} \prod_{i=1}^{m} z_{p_i, k_i}, \tag{4.8}$$

which can be deduced by considering the exponential generating function $\hat{Z}_n(z)$ of the sequence $\{z_{n,k}\}_{k\geq 0}$ and the exponential generating function $\hat{Z}_{\pi,\sigma}(z)$ of the sequence $\{z_k(\pi, \sigma)\}_{k\geq 0}$, that are connected by

$$\hat{Z}_{\pi,\sigma}(z) = \prod_{i=1}^{m} \hat{Z}_{p_i}(z), \tag{4.9}$$

where p_i denotes the number of blocks of π in the i-th block of σ. From this observation it is also verified that the numbers $z_k(\pi, \sigma)$ are only dependent on the m-tupel $(p_1, \ldots, p_m)$, so that also $z_k(p_1, \ldots, p_m)$ could be used as notation for these numbers.

The numbers $z_{n,k}$ obey a recurrence relation.

Theorem 52. Let n, k be positive integers, then the number of chains with repetitions of length k can be calculated by

$$z_{n,k} = \sum_{s_1,\ldots,s_{k-1}\geq 0} \prod_{i=1}^{k-1} S\left(n - \sum_{j=1}^{i-1} s_j, s_i\right) \tag{4.10}$$

or by using the recurrence relation

$$z_{n,k} = \sum_{j=1}^{n} S(n,j) z_{j,k-1} \tag{4.11}$$

with the initial condition $z_{n,0} = \delta_{n,1}$. Note that for $k = 1$ the empty product is assumed to be 1 in the first statement.

Proof. It is easier to reason from the recurrence relation and then to deduce the sum formula by induction over k. Assume that $\hat{0} \leq \rho_1 \leq \ldots \leq \rho_{k-1} \leq \hat{1}$ is a chain of length k. Then ρ_1 is a set partition of $\{1, \ldots, n\}$ with j block, where j ranges from 1 to n and there are $S(n, j)$ ways to choose ρ_1 having these j blocks. Consider the remaining chain $\rho_1 \leq \ldots \leq \hat{1}$. By identifying the j blocks in ρ_1 as singletons there are $z_{j,k-1}$ possible ways to create chains from ρ_1 to $\hat{1}$ and hence

$$z_{n,k} = \sum_{j=1}^{n} S(n,j) z_{j,k-1},$$

as all these n different cases are mutually disjoint and exhaustive.

The initial condition $z_{n,0} = \delta_{n,1}$ holds, as only Π_1 contains a chain of length 0 between $\hat{0}$ and $\hat{1}$, as $\hat{0} = \hat{1}$ holds only for $n = 1$.

The sum formula 4.10 can be derived by induction over k. □

The $z_{n,k}$ can be calculated by the use of the above recurrence relation and the first values of them are found in Table 4.1.

$z_{n,k}$	0	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	0	1	2	3	4	5	6	7
3	0	1	5	12	22	35	51	70
4	0	1	15	60	154	315	561	910
5	0	1	52	358	1304	3455	7556	14532
6	0	1	203	2471	12915	44590	120196	274778
7	0	1	877	19302	146115	660665	2201856	5995892
8	0	1	4140	167894	1855570	11035095	45592666	148154860

Table 4.1.: Chains of length k in Π_n with repetitions between $\hat{0}$ and $\hat{1}$

For some $k \geq 1$ the sequences $\{z_{n,k}\}_{n\geq 0}$ are listed in [28], where $\{z_{n,k}\}_{n\geq 0}$ counts the number of k-level labelled rooted trees with n leaves. The web resource [14] also compiles many aspects about this topic.

For the first values of k one finds the following references in [28].

k=2 [28, **A000110**] k=3 [28, **A000258**] k=4 [28, **A000307**]
k=5 [28, **A000357**] k=6 [28, **A000405**] k=7 [28, **A001669**]
k=8 [28, **A081624**]

According to the chain function η introduced in section 2.3 one may also ask if there is a recurrence relation for the number of chains in Π_n between $\hat{0}$ and $\hat{1}$

of length k without repetitions, that are denoted by $e_{n,k}$. Indeed one may deduce similarly to Theorem 52.

Theorem 53. Let n be a positive integer and k be a non-negative integer, then the numbers $e_{n,k}$ obey the recurrence relation

$$e_{n,k} = \sum_{j=1}^{n-1} S(n, j) e_{j,k-1}. \tag{4.12}$$

with the initial condition $e_{n,0} = \delta_{n,1}$.

Proof. The proof is very similiar to the proof of Theorem 52 and is therefore omitted. □

The first values for the $e_{n,k}$ are found in Table 4.2, [28, **A008826**] and [8], where the last source gives a different interpretation of these numbers. Also note that the $e_{n,k}$ are a special case of the polynomials $c_{n,k}(f_1, \dots, f_{n-k+1})$ introduced in Section 3.7, as

$$e_{n,k} = c_{n,k}(1, \dots, 1) \tag{4.13}$$

holds.

$e_{n,k}$	0	1	2	3	4	5	6
1	1	0	0	0	0	0	0
2	0	1	0	0	0	0	0
3	0	1	3	0	0	0	0
4	0	1	13	18	0	0	0
5	0	1	50	205	180	0	0
6	0	1	201	1865	4245	2700	0
7	0	1	875	16674	74165	114345	56700
8	0	1	4138	155477	1208830	3394790	3919860

Table 4.2.: Chains without repetitions of length k in Π_n between $\hat{0}$ and $\hat{1}$

The Theorem 53 yields a direct computation of the $e_{n,n-1}$, as

$$e_{n,n-1} = \prod_{i=1}^{n-1} S(i+1, i) \tag{4.14}$$

$$= \prod_{i=0}^{n-2} \binom{n-i}{2} \tag{4.15}$$

$$= n\frac{(n-1)!^2}{2^{n-1}} \tag{4.16}$$

holds, whereas for the general numbers $e_{n,k}$ there seems to be no closed formula. A direct combinatorial interpretation of Equation 4.16 as maximal chains between $\hat{0}$ and $\hat{1}$ of the $e_{n,n-1}$ is also readily achieved.

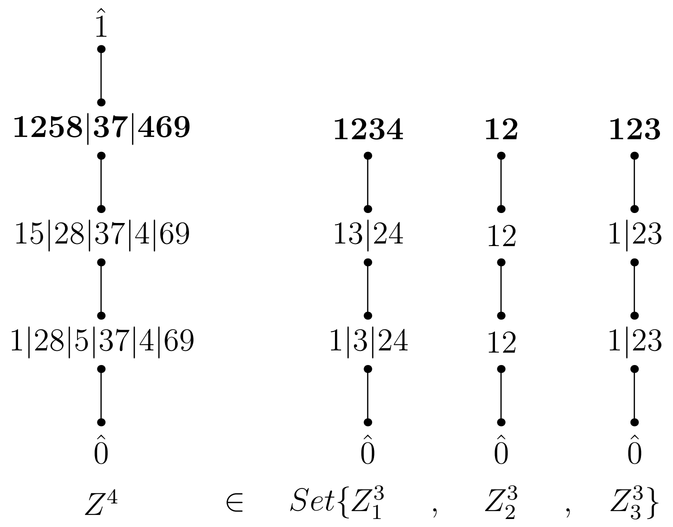

Figure 4.4.: The construction of a chain of length $k+1$ from chains of length k.

It is a surprising fact that even if the numbers $z_{n,k}$ cannot be calculated in a closed form, that the exponential generating function $\hat{Z}_k(z)$ can be derived by recursively applying the exponential formula.

Theorem 54. Let $k \geq 0$ then the exponential generating function $\hat{Z}_k(z)$ of the sequence $\{z_{n,k}\}_{n\geq 0}$ is given by the recursion

$$\hat{Z}_0(z) = z, \tag{4.17}$$
$$\hat{Z}_{k+1}(z) = \exp\left\{\hat{Z}_k(z)\right\} - 1 \qquad \text{for all } k \geq 0. \tag{4.18}$$

Proof. Firstly, denote by $\mathcal{Z}^k$ the labelled class of all chains in the partition lattice Π_n of length k with size given by n between $\hat{0}$ and $\hat{1}$.

Secondly, realize that Π_1 only possesses a chain of length 0, which establishes the start of the induction over k with $k = 0$. Assume that $Z^{k+1} \in \mathcal{Z}^{k+1}$ is an arbitrary chain of length $k+1$ in Π_n

$$Z^{k+1} := \hat{0} \leq \pi^1 \leq \ldots \leq \pi^k \leq \hat{1}$$

and let $\pi^k = \{B_1, \ldots, B_m\}$ with blocksizes $q_i = |B_i|$ respectively. Then Z^{k+1} can be interpreted as an element of the labelled set of m chains $Z_i^k \in \mathcal{Z}^k$ with $1 \leq i \leq m$ of length k in Π_{q_i} or by the notion of the labelled set

$$Z^{k+1} \in \text{Set}\left\{Z_1^k, \ldots, Z_m^k\right\}.$$

The situation is exemplified in Figure 4.4, where a chain of length 4 in Π_9 is constructed out of three chains Z_1^k, Z_2^k, Z_3^k of lengths 3 with $q_1 = 4, q_2 = 2, q_3 = 3$. Realize that here the Set instead of the Seq construction must be used, as the order of the blocks B_i is immaterial. This implies the equation

$$\mathcal{Z}^{k+1} = \mathrm{Set}_{\geq 1}\left\{\mathcal{Z}^k\right\} \quad \text{for all } k \geq 0,$$

leading to the recursion for the exponential generating function $\hat{Z}^k(z)$ of the labelled class $\mathcal{Z}^k$. □

The numbers $z_{n,k}$ and $e_{n,k}$ can be converted into each other leading to the exponential generating function of the sequence $\{e_{n,k}\}_{n\geq 0}$.

Theorem 55. Let $n \geq 1$ and $k \geq 0$ be integers then the numbers $z_{n,k}$ and $e_{n,k}$ obey the equations

$$e_{n,k} = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} z_{n,j}, \tag{4.19}$$

$$z_{n,k} = \sum_{j=0}^{k} \binom{k}{j} e_{n,j} \tag{4.20}$$

and likewise the exponential generating functions of these numbers are connected by

$$\hat{E}_k(z) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \hat{Z}_j(z), \tag{4.21}$$

$$\hat{Z}_k(z) = \sum_{j=0}^{k} \binom{k}{j} \hat{E}_j(z). \tag{4.22}$$

Proof. The theorem is an immediate consequence of Theorem 15. □

An application of Theorem 55 is the construction of the ordinary generating functions $Z^n(z)$ of the sequence $\{z_{n,k}\}_{k\geq 0}$.

Theorem 56. Let n be a positive integer then the ordinary generating function of the sequence $\{z_{n,k}\}_{k\geq 0}$ is given by

$$Z^n(z) = \sum_{j=0}^{n-1} e_{n,j} \frac{z^j}{(1-z)^{j+1}}. \tag{4.23}$$

Proof. By the first identity of Theorem 55 it is

$$z_{n,k} = \sum_{j\geq 0} \binom{k}{j} e_{n,j}.$$

Multiplication with z^k and summing over all k gives

$$\sum_{k\geq 0} z_{n,k} z^k = \sum_{j\geq 0} e_{n,j} \sum_{k\geq 0} \binom{k}{j} z^k.$$

Using the identity $\sum_{k\geq 0}\binom{n+k}{k}z^k = \frac{1}{(1-x)^{n+1}}$ the net result

$$Z^n(z) = \sum_{j=0}^{n-1} e_{n,j}\frac{z^j}{(1-z)^{j+1}}$$

follows. □

Furthermore the Theorem 52 could be facilitated to find a recurrence relation for the $Z^n(z)$.

Theorem 57. Let $Z^n(z)$ be the ordinary generating function of the sequence $\{z_{n,k}\}_{k\geq 0}$, then for all $n \geq 2$ the recurrence relation

$$Z^n(z) = \frac{z}{1-z}\sum_{j=1}^{n-1} S(n,j)Z^j(z) \tag{4.24}$$

is valid, where $Z^1(z) = \frac{1}{1-z}$ is used as initial condition.

Proof. By the recurrence relation proven in Theorem 52 one gets after multiplication with z^k and summing over all k

$$Z^n(z) - z_{n,0} = zZ^n(z) + z\sum_{j=1}^{n-1} S(n,j)Z^j(z)$$

$$Z^n(z) = \frac{z_{n,0}}{1-z} + \frac{z}{1-z}\sum_{j=1}^{n-1} S(n,j)Z^j(z),$$

so that the claim is proven, as $z_{n,0} = \delta_{n,0}$ holds. □

Using the Theorem 57 the first ordinary generating functions $Z^n(z)$ are given by

$$Z^1(z) = \frac{1}{1-z} \tag{4.25}$$

$$Z^2(z) = \frac{z}{(1-z)^2} \tag{4.26}$$

$$Z^3(z) = \frac{z(1+2z)}{(1-z)^3} \tag{4.27}$$

$$Z^4(z) = \frac{z(1+11z+6z^2)}{(1-z)^4} \tag{4.28}$$

$$Z^5(z) = \frac{z(1+47z+108z^2+24z^3)}{(1-z)^5} \tag{4.29}$$

$$Z^6(z) = \frac{z(1+197z+1268z^2+1114z^3+120z^4)}{(1-z)^6} \tag{4.30}$$

$$Z^7(z) = \frac{z(1+870z+13184z^2+29383z^3+12542z^4+720z^5)}{(1-z)^7} \tag{4.31}$$

$$Z^8(z) = \frac{z(1+4132z+134802z^2+628282z^3+659797z^4+155546z^5+5040z^6)}{(1-z)^8}. \tag{4.32}$$

It is intriguing at the first encounter that the sign alternating sum $\sum_{j\geq 0}(-1)^j e_{n,j}$ obeys an easy representation.

Theorem 58. Let $n \geq 1$ then

$$\sum_{j=0}^{n-1}(-1)^j e_{n,j} = (-1)^{n-1}(n-1)! \tag{4.33}$$

holds.

Proof. Firstly, realize that the exponential generating function $\hat{Z}^k(z)$of the sequence $\{e_{n,k}\}_{n\geq 0}$ satisfies $\hat{Z}^k(z) = f^{<k>}(z)$ with $f^{<k>}(z)$ being the k-fold iterate of the function

$$f(z) = \exp z - 1 = \sum_{k\geq 1} 1\frac{z^k}{k!}. \tag{4.34}$$

This is an immediate consequence of Theorem 54. Secondly, juxtapose the two statements found in Theorem 55 and Equation 3.131.

$$f_n^{<k>} = \sum_{j=0}^{n-1}\binom{k}{j}c_{n,j}(f_1,\ldots,f_{n-j+1}) \tag{4.35}$$

$$z_{n,k} = \sum_{j=0}^{n-1}\binom{k}{j}e_{n,j} \tag{4.36}$$

implying

$$f_n^{<k>} = \sum_{j=0}^{n-1}\binom{k}{j}e_{n,j}. \tag{4.37}$$

Setting $k = -1$ leads to

$$f_n^{<-1>} = \sum_{j=0}^{n-1}(-1)^j e_{n,j}. \tag{4.38}$$

On the other hand $f^{<-1>}(z) = \ln(1+z)$ is the compositional inverse of $f(z)$, so that

$$f^{<-1>}(z) = \sum_{n\geq 1} f_n^{<-1>}\frac{z^n}{n!} \tag{4.39}$$

$$= \sum_{n\geq 1}(-1)^{n-1}(n-1)!\frac{z^n}{n!} \tag{4.40}$$

follows. Comparing the coefficients of the Equation 4.40 with 4.38 proves the claim. □

There is little known about the asymptotic growth of the overall number of chains in the partition lattice between $\hat{0}$ and $\hat{1}$, but some efforts had been made by [12].

If E_n denotes the number of all chains in the partition lattice Π_n without repetitions between $\hat{0}$ and $\hat{1}$ the recurrence relation $E_n = \sum_{k=1}^{n-1} S(n,k)E_k$ with $E_1 = 1$ is readily established. It is notable that the exponential generating function $\hat{E}(z)$ for the sequence $\{E_n\}_{n\geq 0}$ satisfies the interesting functional equation

$$\hat{E}(z) = \frac{1}{2}\hat{E}\left(e^z - 1\right) + \frac{z}{2}, \tag{4.41}$$

as can be seen by

$$\begin{aligned}
\frac{1}{2}\hat{E}\left(e^z - 1\right) + \frac{z}{2} &= \frac{1}{2}\sum_{k\geq 0} E_k \frac{(e^z-1)^k}{k!} + \frac{z}{2} && (4.42)\\
&= \frac{1}{2}\sum_{k\geq 0} E_k \sum_{n\geq 0} S(n,k)\frac{z^n}{n!} + \frac{z}{2} && (4.43)\\
&= \frac{1}{2}\sum_{n\geq 0}\left(\sum_{k=0}^{n} S(n,k)E_k\right)\frac{z^n}{n!} + \frac{z}{2} && (4.44)\\
&= \frac{1}{2}\sum_{n\geq 2} 2E_n\frac{z^n}{n!} + E_1 z && (4.45)\\
&= \hat{E}(z). && (4.46)
\end{aligned}$$

It is shown in [12], that the asymptotic growth $\tilde{E}_n$ of the numbers E_n is

$$E_n \sim \Lambda \frac{n!^2}{(2\ln 2)^n n^{1+(\ln(2))/3}} = \tilde{E}_n, \tag{4.47}$$

where Λ is a constant factor, which is approximately 1.0986858055. The number Λ is known under Lengyel's constant in mathematics. The accuracy of the approximation in Equation 4.47 is exemplified in Table 4.3.

n	E_n	$\tilde{E}_n$	$E_n/\tilde{E}_n$
1	1	0.792534282	1.26177507
2	1	0.974181546	1.02650271
3	4	3.839281644	1.04186157
4	32	31.09630476	1.02906118
5	436	426.0811837	1.02327917
6	9012	8840.226674	1.01943087
7	262760	258457.5611	1.01664660
8	10270696	10123316.79	1.01455839
9	518277560	511659597.3	1.01293431
10	3.279592802e+10	3.241872663e+10	1.01163529
15	5.029959425e+20	4.991313475e+20	1.00774264
25	1.435009706e+45	1.428386442e+45	1.00463688
50	6.664062828e+119	6.648673331e+119	1.00231467
75	7.643892340e+205	7.632121952e+205	1.00154222
100	2.156447666e+299	2.153957005e+299	1.00115632
125	1.894503998e+398	1.892753410e+398	1.00092489
150	3.960967570e+501	3.957917400e+501	1.00077065

Table 4.3.: Asymptotic growth of the number of chains without repetitions in Π_n

4.2. The Möbius function in Π_n

It is remarkable that a closed formula for the Möbius function $\mu(\pi, \sigma)$ with $\pi, \sigma \in \Pi_n$ can be deduced by considering the chains without repetitions in Π_n between $\hat{0}$ and $\hat{1}$.

The combination of the theorems 58, 51 and 13 leads to the announced formula for the Möbius function $\mu(\pi, \sigma)$.

Theorem 59. Let $\pi, \sigma \in \Pi_n$ then $\mu(\pi, \sigma)$ is

$$\mu(\pi, \sigma) = \begin{cases} (-1)^{|\pi|-|\sigma|} \prod_{C_i \in \sigma}(p_i - 1)! & \pi \leq \sigma \\ 0 & \text{else} \end{cases}, \tag{4.48}$$

where p_i denotes the number of blocks of π that are in the i-th block of σ.

Proof. Let $\pi, \sigma \in \Pi_n$ then the interval $[\pi, \sigma]$ is by Theorem 51 isomorphic to some product order $\coprod_{i=1}^{|\sigma|} \Pi_{p_i}$, so that due to Theorem 13

$$\mu(\pi, \sigma) = \prod_{i=1}^{|\sigma|} \mu_{p_i}(\hat{0}, \hat{1})$$

holds, where μ_{p_i} denotes the Möbius function in Π_{p_i}.

Utilizing the Theorem 58 yields

$$\mu_{p_i}(\hat{0}, \hat{1}) = (-1)^{p_i - 1}(p_i - 1)!$$

implying

$$\mu(\pi,\sigma) = \prod_{i=1}^{|\sigma|} (-1)^{p_i-1}(p_i-1)!$$
$$= (-1)^{|\pi|-|\sigma|} \prod_{i=1}^{|\sigma|} (p_i-1)!,$$

which proves the claim. □

There is an interesting connection between the partial Bell polynomials and the Möbius inversion in the partition lattice, when restricting f and g to multiplicative functions.

Let $f\colon \Pi_n \to \mathbb{C}$ be a function on the partition lattice Π_n, so that for all $\pi \in \Pi_n$ having k_i blocks of size i

$$f(\pi) = f(1^{k_1} \dots n^{k_n}) = \prod_{i=1}^{n} f_i^{k_i} \tag{4.49}$$

holds for given f_i then f is called *multiplicative* on Π_n. Assume that $f, g\colon \Pi_n \to \mathbb{C}$ with f being multiplicative, so that

$$f(\pi) = \sum_{\sigma \le \pi} g(\sigma) \tag{4.50}$$
$$g(\pi) = \sum_{\sigma \le \pi} f(\sigma)\mu(\sigma,\pi) \tag{4.51}$$

holds by Theorem 2.3 for all $\pi \in \Pi_n$. Under this assumptions the equations 4.50 and 4.51 can be expressed by the Bell polynomials and the logarithmic Bell polynomials.

Theorem 60. Let $\pi \in \Pi_n$ having k_i blocks of size i then Equation 4.51 and 4.50 with f assumed to be multiplicative are given by

$$g(\pi) = g(1^{k_1} \dots n^{k_n}) = \prod_{i=1}^{n} L_i(f_1,\dots,f_i)^{k_i} \tag{4.52}$$
$$f(\pi) = f(1^{k_1} \dots n^{k_n}) = \prod_{i=1}^{n} B_i(g_1,\dots,g_i)^{k_i}, \tag{4.53}$$

where L_n denotes the logarithmic Bell polynomials and B_n the complete Bell polynomials. Note that Equation 4.52 implies the multiplicativity of g with g_i equals to $L_i(f_1,\dots,f_i)$.

Proof. Firstly, examine the Equation 4.51 together with Theorem 59. The multiplicativity of f implies

$$\begin{aligned} g(\pi) &= \sum_{\sigma\leq\pi} f(\sigma)\mu(\sigma,\pi) \\ &= \prod_{i=1}^{n}\left(\sum_{r=1}^{i}(-1)^{r-1}(r-1)! \sum_{\substack{s_1+\ldots is_i=i \\ s_1+\ldots+s_i=r}} \frac{i!}{s_1!\cdots s_i!1!^{s_1}\cdots i!^{s_i}} f_1^{s_1}\cdots f_i^{s_i}\right)^{k_i} \\ &= \prod_{i=1}^{n} L_i(f_1,\ldots,f_i)^{k_i}. \end{aligned}$$

Secondly, the Equation 4.51 together with the multiplicativity of g implies

$$\begin{aligned} f(\pi) &= \sum_{\sigma\leq\pi} g(\sigma) \\ &= \prod_{i=1}^{n}\left(\sum_{r=1}^{i} \sum_{\substack{s_1+\ldots is_i=i \\ s_1+\ldots+s_i=r}} \frac{i!}{s_1!\cdots s_i!1!^{s_1}\cdots i!^{s_i}} g_1^{s_1}\cdots g_i^{s_i}\right)^{k_i} \\ &= \prod_{i=1}^{n} B_n(g_1,\ldots,g_n)^{k_i}. \end{aligned}$$

□

4.3. Sums in Π_n

In this last section different representations of the sums

$$R(n,r) = \sum_{\pi\in\Pi_n} |\pi|^{\underline{r}} \tag{4.54}$$

$$T(n,r) = \sum_{\pi\in\Pi_n} |\pi|^{r} \tag{4.55}$$

are considered.

The numbers $R(n,r)$ count the number of ways to select a partition $\pi \in \Pi_n$ and then to select r distinct blocks in this partition regarding the order of the selection of these blocks. The numbers $T(n,r)$ are only differing from the numbers $R(n,r)$ as the r selected blocks are not constrained to be distinct.

A first simplification of the sums is an easy task, as there are $S(n,k)$ partitions in Π_n with k blocks, so that

$$R(n,r) = \sum_{k=1}^{n} S(n,k)k^{\underline{r}}, \tag{4.56}$$

$$= r! \sum_{k=1}^{n} S(n,k)\binom{k}{r}, \tag{4.57}$$

$$T(n,r) = \sum_{k=1}^{n} S(n,k)k^{r} \tag{4.58}$$

holds.

It is possible to deduce an interesting recurrence relation for the $R(n,r)$. But the author was not able to find a combinatorial interpretation of it.

Theorem 61. Let n, r be non-negative integers then the numbers $R(n,r)$ satisfy the recurrence relation

$$R(n,r) = R(n+1, r-1) - rR(n, r-1) - (r-1)R(n, r-2) \tag{4.59}$$

with the boundary values

$$R(n,0) = B(n) \tag{4.60}$$

$$R(n,r) = 0 \quad r < 0. \tag{4.61}$$

Proof. The recurrence relation can be derived algebraically

$$\begin{aligned}
R(n,r) &:= \sum_{k=1}^{n} k^{\underline{r-1}} kS(n,k) - (r-1)\sum_{k=1}^{n} k^{\underline{r-1}} S(n,k) \\
&= \sum_{k=1}^{n+1} k^{\underline{r-1}} S(n+1,k) - \sum_{k=1}^{n} (k+1)^{\underline{r-1}} S(n,k) - (r-1)R(n,r-1) \\
&= R(n+1,r-1) - (r-1)R(n,r-1) - \sum_{k=1}^{n} k^{\underline{r-1}} S(n,k) - (r-1)\sum_{k=1}^{n} k^{\underline{r-2}} S(n,k) \\
&= R(n+1,r-1) - rR(n,r-1) - (r-1)R(n,r-2),
\end{aligned}$$

applying the identities (3.5) and (3.17). □

Another recurrence relation for the numbers $R(n,r)$ can be deduced by using a purely combinatorial argument which resembles Theorem 18.

Theorem 62. Let n be a non-negative integer and r be a positive integer then the numbers $R(n,r)$ satisfy the following recurrence relation

$$R(n,r) = \sum_{j=0}^{n-1} \binom{n-1}{j} [R(n-j-1, r) + rR(n-j-1, r-1)] \tag{4.62}$$

with the boundary condition $R(n,0) = B(n)$.

Proof. The number of partitions of $X = \{1, \dots, n\}$ with exactly r marked blocks not regarding the order of the marking is given by $R(n,r)/r!$. The element n can be either in a marked or in a non marked block. Furthermore the element n must be joined by j other elements from the set $\{1, \dots n-1\}$. Hence the recurrence relation follows. □

The first values for $R(n,r)/r!$ are listed in Table 4.4 and [28, **A049020**].

	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	2	3	1					
3	5	10	6	1				
4	15	37	31	10	1			
5	52	151	160	75	15	1		
6	203	674	856	520	155	21	1	
7	877	3263	4802	3556	1400	287	28	1
8	4140	17007	28337	24626	11991	3290	490	36
9	21147	94828	175896	174805	101031	34671	6972	786
10	115975	562595	1146931	1279240	853315	350889	88977	13620

Table 4.4.: Number of partitions in Π_n with r marked blocks

Substituting the first six values for r in the recurrence relation of Theorem 61 yields a representation of the $R(n,r)$ as a linear combination of Bell numbers

$$R(n,0) = B(n) \tag{4.63}$$
$$R(n,1) = B(n+1) - B(n) \tag{4.64}$$
$$R(n,2) = B(n+2) - 3B(n+1) + B(n) \tag{4.65}$$
$$R(n,3) = B(n+3) - 6B(n+2) + 8B(n+1) - B(n). \tag{4.66}$$

If the coefficients $b_{r,k}$ in this linear combinations are defined by

$$R(n,r) = \sum_{k=0}^{r} b_{r,k} B(n+r-k), \tag{4.67}$$

then the triangle of numbers $b_{r,k}$ in Table 4.5 is generated, that is also found in [28, **A046716**] when omitting the signs.

Besides the already given linear combination of the numbers $R(n,r)$ as a linear combination of Bell numbers there is another interpretation.

Theorem 63. The numbers $R(n,r)$ can be represented by the following linear combination of Bell numbers

$$R(n,r) = \sum_{k=0}^{n} \binom{n}{k} r! S(k,r) B(n-k). \tag{4.68}$$

	k=0	1	2	3	4	5	6	7	8	9	10	11
r=0	1											
1	1	-1										
2	1	-3	1									
3	1	-6	8	-1								
4	1	-10	29	-24	1							
5	1	-15	75	-145	89	-1						
6	1	-21	160	-545	814	-415	1					
7	1	-28	301	-1575	4179	-5243	2372	-1				
8	1	-36	518	-3836	15659	-34860	38618	-16072	1			
9	1	-45	834	-8274	47775	-163191	318926	-321690	125673	-1		
10	1	-55	1275	-16290	125853	-606417	1809905	-3197210	2995011	-1112083	1	
11	1	-66	1870	-29865	296703	-1908060	8002742	-21474255	34975061	-30840304	10976184	-1

Table 4.5.: Triangle of the numbers $b_{r,k}$

Proof. The number $R(n, r)$ counts the number of possible ways to distribute n distinguishable balls into n indistinguishable and r distinguishable boxes, regardless of the order of the balls in the boxes, fulfilling the restriction that the r boxes must contain at least one ball.

The set of possible distributions can be divided into $n + 1$ disjoint subsets due to the number of balls that are distributed among the r distinguishable boxes. Assume that k balls are distributed into the r and $n - k$ balls are distributed into the n boxes. There are $\binom{n}{k}$ possible ways to choose k from the n balls and $S(k, r)r!$ possible ways to distribute these k balls among the r boxes, so that no box remains empty and $B(n - k)$ possible ways to distribute the remaining $n - k$ balls into the n boxes in an arbitrary fashion. □

As a consequence of Theorem 63 it is possible to derive the bivariate exponential generating function for the $R(n, k)$.

Theorem 64. Let $\hat{R}(z, u)$ be the bivariate exponential generating function

$$\hat{R}(z, u) = \sum_{n\geq 0, k\geq 0} R(n, k)\frac{z^n}{n!}\frac{u^k}{k!} \tag{4.69}$$

then

$$\hat{R}(z) = \exp\{(\exp z - 1)(u + 1)\}. \tag{4.70}$$

Proof. Utilizing the exponential generating functions in equations 3.95 and 3.96 one deduces

$$\begin{aligned}
\exp\{(\exp z - 1)(u + 1)\} &= \exp\{\exp z - 1\}\exp\{u(\exp z - 1)\} \\
&= \sum_{n\geq 0} B(n)\frac{z^n}{n!}\sum_{n\geq 0, k\geq 0} S(n, k)u^k\frac{z^n}{n!} \\
&= \sum_{n\geq 0}\left(\sum_{k\geq 0} u^k \sum_{i=0}^{n}\binom{n}{i}S(i, k)B(n - i)\right)\frac{z^n}{n!} \\
&= \sum_{n\geq 0, k\geq 0} R(n, k)\frac{u^k}{k!}\frac{z^n}{n!},
\end{aligned}$$

where the last conclusion was drawn under the use of Theorem 63. □

According to Theorem 64 the following three exponential generating functions are defined in analogy to equations 3.95, 3.96, and 3.97 for the Stirling numbers of the second kind

$$\hat{R}(z) = \sum_{n\geq 0} 2^n B(n)\frac{z^n}{n!} = \exp\{2(\exp z - 1)\} = \hat{R}(z, 1), \tag{4.71}$$

$$\hat{R}_n(u) = \sum_{k\geq 0} R(n, k)\frac{u^k}{k!} = n!\,[z^n]\,\hat{R}(z, u), \tag{4.72}$$

$$\hat{R}^k(z) = \sum_{n\geq 0} R(n, k)\frac{z^n}{n!} = \exp\{\exp z - 1\}\frac{1}{k!}(\exp z - 1)^k = k!\left[u^k\right]\hat{R}(z, u). \tag{4.73}$$

The Theorem 64 could be also interpreted in the light of the results in Section 3.5 using the framework of [13].

Theorem 65. The labelled class $\mathcal{R}$ of all marked set partitions satisfies the equation

$$\mathcal{R} = \mathrm{Set}\{\mathrm{Set}_{k\geq 1}\{\mathcal{Z}_r\}\} \star \mathrm{Set}\{\mathrm{Set}_{k\geq 1}\{\mathcal{Z}_b\}\}, \tag{4.74}$$

where $\mathcal{Z}_r$ and $\mathcal{Z}_b$ are disjoint atomic labelled classes.

Proof. Every marked set partition $\pi \in \mathcal{R}$ can be interpreted as a set partition containing red and blue blocks. The blue blocks are taken as the marked blocks of the partition, whereas the red blocks represent the unmarked ones.

The left-hand side of the labelled product in Equation 4.74 represents the construction of all partitions having only red blocks and the right-hand side the blue ones. The labelled product of both classes then distributes the labels in all allowed possible ways. □

Using v as a marking variable for the number of blocks for $\mathrm{Set}\{\mathrm{Set}_{k\geq 1}\{\mathcal{Z}_r\}\}$ and u as a marking variable for the number of blocks for $\mathrm{Set}\{\mathrm{Set}_{k\geq 1}\{\mathcal{Z}_b\}\}$ the exponential generating function $\hat{R}(z; u, v)$ of the class $\mathcal{R}$ is given by

$$\hat{R}(z; u, v) = \exp\{v(e^z - 1)\} \exp\{u(e^z - 1)\}. \tag{4.75}$$

Setting $v = 1$ results in the already found exponential generating function in Theorem 64.

Note that Equation 4.72 is not stated in an amendable form, but using Theorem 62 it is possible to derive a recurrence relation for the $\hat{R}_n(z)$.

Theorem 66. Let n be a non-negative integer and $\hat{R}_n(u)$ the exponential generating function of the sequence $\{R(n,k)\}_{k\geq 0}$ stated in Equation 4.72 then

$$\hat{R}_n(u) = (1+u)\sum_{j=0}^{n-1}\binom{n-1}{j}\hat{R}_{n-j-1}(u) \text{for all } n \geq 1 \tag{4.76}$$

holds with the boundary condition $\hat{R}_0(u) = 1$.

Proof. Just utilize Theorem 62 to derive this result. □

The further aim is to find a more explicit form for the occurring coefficients of the linear combination in Equation 4.67. It is possible to attack this problem in different ways that are subsequently presented. Firstly, a combinatorial argument in the language of set partitions is used and secondly, a graph theoretic approach is discussed utilizing the chromatic polynomial of a special graph.

The next theorem transforms the numbers $R(n, r)$ in a counting problem of certain set partitions.

Theorem 67. Let $\mathscr{R}(n, r)$ be the subset of set partitions of Π_{n+r} satisfying the following two constraints.

1. For every set partition $\pi \in \mathscr{R}(n, r)$ there is no block containing two different elements from the set $\{n + 1, \ldots n + r\}$.

2. For every set partition $\pi \in \mathscr{R}(n,r)$ there is no singleton block containing an element from the set $\{n+1,\ldots,n+r\}$.

Then the cardinality of $\mathscr{R}(n,r)$ equals $R(n,r)$.

Proof. Let $n = 5$ and $r = 3$ then the partition 15|**68**|234**7** is failing to satisfy the first constraint and the partition 123|**6**|**7**|45**8** is failing to satisfy the second one. Every allowed partition in Π_{n+r} is corresponding to exactly one ordered selection of r blocks of a partition in Π_n. The elements $\{n+1,\ldots,n+r\}$ are used as markers, so that that the i-th selected block is corresponding to the block containing the element $n+i$ or using an example the partition 12**6**|3**7**|45**8** $\in \mathscr{R}(5,3)$ is corresponding to the ordered selection $(\{1,2\},\{3\},\{4,5\})$. Using the above examples the bijection between the two sets becomes clear. The technical details of the bijection are omitted. □

Due to Theorem 67 it is therefore possible to count certain set partitions of Π_{n+r} to determine the numbers $R(n,r)$. The theorems 70 and 74 utilize this fact by application of the Inclusion-Exclusion principle. In order to state this theorems a few definitions had to be made.

Firstly, let $\mathscr{M}(n,r)$ be the subset of Π_{n+r} satisfying merely the first condition of Theorem 67 and let $M(n,r)$ be the cardinality of this set. Secondly, let $\mathscr{A}_{(i,j)}$ be the subset of Π_{n+r}, so that $\pi \in \mathscr{A}_{(i,j)}$ iff $n+i \sim_\pi n+j$ for all $1 \le i < j \le r$. And at least define by $\mathscr{B}_i \subseteq \mathscr{M}(n,r)$ and $\mathscr{C}_i \subseteq \Pi_{n+r}$ the set of set partitions, so that $\{n+i\}$ is a singleton block.

In analogy to Theorem 63 it is possible to deduce a similar linear combination for the $M(n,r)$, when dropping the second constraint in Theorem 67.

Theorem 68. The numbers $M(n,r)$ satisfy

$$M(n,r) = \sum_{k=0}^{n} \binom{n}{k} B(n-k) r^k. \tag{4.77}$$

Proof. Argue in the same vein like in theorem 63 with keeping in mind that it is not longer necessary that every of the distinguishable r boxes must contain at least one element, so that the factor $S(k,r)$ is replaced by r^k. □

Following the ideas of Theorem 64 it is also possible to derive the exponential generating function for the $M(n,r)$.

Theorem 69. Let $\hat{M}(z,u)$ be the bivariate exponential generating function

$$\hat{M}(z,u) = \exp\{e^z(u+1) - 1\} \tag{4.78}$$

then

$$\hat{M}(z,u) = \sum_{n\ge0,k\ge0} M(n,k) \frac{u^k}{k!} \frac{z^n}{n!} \tag{4.79}$$

holds.

Proof. Consider

$$\begin{aligned}
\exp\{e^z(u+1)-1\} &= \exp\{e^z-1\}\exp\{ue^z\}\\
&= \sum_{n\geq 0} B(n)\frac{z^n}{n!}\sum_{k\geq 0}\frac{u^k e^{kz}}{k!}\\
&= \sum_{n\geq 0} B(n)\frac{z^n}{n!}\sum_{k\geq 0}\frac{u^k}{k!}\sum_{m\geq 0}k^m\frac{z^m}{m!}\\
&= \sum_{n\geq 0,k\geq 0}\left(\sum_{i=0}^{n}\binom{n}{i}B(n-i)k^i\right)\frac{u^k}{k!}\frac{z^n}{n!}\\
&= \sum_{n\geq 0,k\geq 0} M(n,k)\frac{u^k}{k!}\frac{z^n}{n!}
\end{aligned}$$

where the last equivalence follows from Theorem 68. □

Again three exponential generating function can be deduced from $\hat{M}(z,u)$

$$\hat{M}(z) = \exp\{2e^z - 1\} = \hat{M}(z,1), \tag{4.80}$$

$$\hat{M}_n(z) = \sum_{k\geq 0} M(n,k)\frac{u^k}{k!} = n!\,[z^n]\,\hat{M}(z,u), \tag{4.81}$$

$$\hat{M}^k(z) = \sum_{n\geq 0} M(n,k)\frac{z^n}{n!} = \exp\{e^z + kz - 1\} = k!\left[u^k\right]\hat{M}(z,u). \tag{4.82}$$

After examining questions of exponential generating functions of the sets $\mathscr{R}(n,r)$ and $\mathscr{M}(n,r)$ the primal question of the linear combination 4.67 of the $R(n,r)$ is considered.

Theorem 70. The numbers $M(n,r)$ are given by the sum

$$\begin{aligned}
M(n,r) &= \sum_{k=0}^{r} B(n+k)\left(m^e_{r,k} - m^o_{r,k}\right) &(4.83)\\
&= \sum_{k=0}^{r} B(n+k)m_{r,k}, &(4.84)
\end{aligned}$$

where the $m^e_{r,k}$,$m^o_{r,k}$ are counting the number of partitions in Π_r with exactly k blocks that are a supremum of an even/odd number of atoms of Π_r, respectively and furthermore $m_{r,k} = m^e_{r,k} - m^o_{r,k}$.

Proof. Define the set $J = \{(i,j) : 1 \leq i < j \leq r\}$ and recognize the identity

$$\mathscr{M}(n,r) = \bigcap_{(i,j)\in J}\overline{\mathscr{A}_{(i,j)}},$$

which allows the application of the Inclusion-Exclusion principle, so that

$$M(n,r) = \sum_{I\subseteq J}(-1)^{|I|}\left|\bigcap_{(i,j)\in J}\mathscr{A}_{(i,j)}\right|$$

holds. Then take the following identity into account

$$\bigcap_{(i,j)\in J} \mathscr{A}_{(i,j)} = \mathfrak{F}\left(\sup_{(i,j)\in J} \pi_{n+i,n+j}\right),$$

which implies

$$\left|\bigcap_{(i,j)\in J} \mathscr{A}_{(i,j)}\right| = B(|\sup_{(i,j)\in J} \pi_{n+i,n+j}|)$$

as $\mathfrak{F}(\pi) \simeq \Pi_{|\pi|}$ and realize that $|\sup_{(i,j)\in J} \pi_{n+i,n+j}| \in \{n+1,\dots,n+r\}$ as $\sup_{(i,j)\in J} \pi_{n+i,n+j}$ always contains the singleton blocks $\{k\}$ for $1 \le k \le n$, so that the representation of the numbers $M(n,r)$

$$M(n,r) = \sum_{k=0}^{r} B(n+k) \sum_{j=0}^{\binom{r}{2}} (-1)^j m_{r,k}^j$$

is found, where the $m_{r,k}^j$ are counting the number of set partitions of Π_r with exactly k blocks, that are a supremum of exactly j atomic elements. By setting

$$m_{r,k}^e = \sum_{j \text{ even}} m_{r,k}^j \qquad\qquad m_{r,k}^o = \sum_{j \text{ odd}} m_{r,k}^j$$

the proof of the claim follows. □

It is quite striking that even though the numbers $m_{r,k}^j$ are complicated to determine, that this does not hold for the numbers $m_{r,k}$ of Theorem 70.

Theorem 71. Let $s(r,k)$ denote the Stirling numbers of the first kind, then

$$m_{r,k} = s(r,k). \tag{4.85}$$

Proof. Considering the connection of graphs and set partitions described in Section 3.2 and in Chapter 4 in connection with the Equation 3.42 as a result of Theorem 34 the desired result follows. □

Corollary 72. The numbers $M(n,r)$ are given by the sum

$$M(n,r) = \sum_{k=0}^{r} s(r,k)B(n+k) \tag{4.86}$$

as a linear combination of Bell numbers and moreover the identity

$$\sum_{k=0}^{r} s(r,k)B(n+k) = \sum_{k=0}^{n} \binom{n}{k} B(n-k) r^k \tag{4.87}$$

holds.

The numbers $M(n,r)$ can be interpreted as a evaluation of $M(\overline{K_n} \cup K_r, z)$, where $\overline{K_n} \cup K_r$ is the disjoint union of the empty graph $\overline{K_n}$ with n vertices and the complete graph K_r with r vertices. Using $\chi(\overline{K_n} \cup K_r, z) = z^n z^{\underline{r}} = \sum_k s(r,k) z^{k+n}$ and Theorem 31 another proof for Theorem 71 is found, by

$$M(n,r) = M(\overline{K_n} \cup K_r, 1) \tag{4.88}$$

$$= \sum_{k=0}^{r} s(r,k) B(n+k). \tag{4.89}$$

Moreover the polynomial $M(G,z)$ serves as a generalization for the numbers $M(n,r)$.

It is notable, but with the observations already made quite trivial, that the sum of the coefficients $m_{r,k} := s(r,k)$ of the linear combination must vanish for all $r \geq 2$.

Theorem 73. Let $M(n,r)$ with $r \geq 2$ given by

$$M(n,r) = \sum_{k=0}^{r} B(n+k) s(r,k), \tag{4.90}$$

then the sum of the coefficients $s(r,k)$ vanishes.

Proof. The first proof of this fact could be readily deduced from theorem 24, as

$$\sum_{k=0}^{r} s(r,k) S(k,1) = \sum_{k=0}^{r} s(r,k) = \delta_{r,1} \tag{4.91}$$

holds.

Another interesting proof arrives from Lemma 32 that proved already useful in the proof of theorem 33.

Consider the sum

$$\sum_{k=0}^{r} m_{r,k} = \sum_{k=0}^{r} m^e_{r,k} - m^o_{r,k} = m^e_r - m^o_r, \tag{4.92}$$

where m^e_r and m^o_r denotes the number of graphs with r vertices having an even and odd number of edges.

The set E of all possible edges of a graph with r vertices contains exactly $\binom{r}{2}$ elements. So that for all $r \geq 2$ it is $E \neq \emptyset$. As $E \neq \emptyset$ it contains exactly $2^{|E|-1}$ subsets of even and of odd size, which proves the claim. □

With the previous conclusions it is possible to get a more explicit representation of the $R(n,r)$. Just recall the definition of the set $\mathscr{B}_i \subset \mathscr{M}(n,r)$.

Theorem 74. The numbers $R(n,r)$ and $M(n,r)$ are connected by the relations

$$R(n,r) = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} M(n,k), \tag{4.93}$$

$$M(n,r) = \sum_{k=0}^{r} \binom{r}{k} R(n,k). \tag{4.94}$$

Proof. Let $J = \{1, \dots, r\}$ then $\mathscr{R}(n,r)$ can be described in terms of the sets $\mathscr{B}_i$,

$$\mathscr{R}(n,r) = \bigcap_{j \in J} \overline{\mathscr{B}_j},$$

which allows again the application of the Inclusion-Exclusion principle

$$R(n,r) = \sum_{I \subseteq J} (-1)^{|I|} \left| \bigcap_{i \in I} \mathscr{B}_i \right|.$$

Note that the cardinality of $\left|\bigcap_{i\in I} \mathscr{B}_i\right|$ depends only on the cardinality of I, as

$$\left| \bigcap_{i \in I} \mathscr{B}_i \right| = M(n, r - |I|)$$

holds, so that

$$R(n,r) = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} M(n,k)$$

the first claim is proven. The second relation accounts for

$$M(n,r) = \sum_{I \subseteq J} \left| \bigcap_{i \in I} \overline{\mathscr{B}_i} \right|$$

implying

$$M(n,r) = \sum_{k=0}^{r} \binom{r}{k} R(n,k)$$

the second claim. □

The Corollary 75 is a direct consequence of Theorem 74.

Corollary 75. The numbers $R(n,r)$ obey the linear combination

$$R(n,r) = \sum_{k=0}^{r} B(n + r - k) b_{r,k} \tag{4.95}$$

in terms of Bell numbers with

$$b_{r,k} = \sum_{j=r-k}^{r} (-1)^{r-j} \binom{r}{j} s(j, r-k). \tag{4.96}$$

The sum $R(n,r)$ can be generalized to range over an interval $[\sigma, \pi]$ in Π_n.

Definition 76. Let $\pi, \sigma \in \Pi_n$ with $\pi \leq \sigma$ and $I = [\pi, \sigma]$ then define

$$R(I, r) = \sum_{\rho \in I} |\rho|^{\underline{r}}. \tag{4.97}$$

The generalized sums $R(I, r)$ are conveyed by the multiplication of the exponential generating functions $\hat{R}_n(z)$.

Theorem 77. Let $I = [\sigma, \pi]$ be an interval in Π_n, so that $|\pi| = k$ and assume that p_i blocks of σ are in the i-th block of π, then

$$R(I, r) = \sum_{\substack{r_1 + \ldots + r_k = r \\ r_i \geq 0}} \binom{r}{r_1, \ldots, r_k} \prod_{i=1}^{k} R(p_i, r_i) \tag{4.98}$$

holds.

Proof. Let $\hat{R}_n(z)$ be the exponential generating function defined by Equation 4.72 and $\hat{R}_I(z)$ the exponential generating function

$$\hat{R}_I(z) = \sum_{r \geq 0} R(I, r) \frac{z^r}{r!} \tag{4.99}$$

then $\hat{R}_I(z) = \prod_{i=1}^{k} \hat{R}_{p_i}(z)$ holds. Extracting the coefficients of $\frac{z^r}{r!}$ proves the desired claim. □

The last theorem assumes a very simple form if $r = 1$ is stipulated.

Corollary 78. Let $I = [\sigma, \pi]$ be an interval in Π_n with $|\pi| = k$ and assume that p_i blocks of σ are in the i-th block of π then

$$R(I, 1) = \sum_{j=1}^{k} \Big(B(p_j + 1) - B(p_j) \Big) \prod_{\substack{i=1 \\ i \neq j}}^{k} B(p_i) \tag{4.100}$$

holds.

It is also possible to specialize the Theorem 77 in a different manner under the assumption, that there are v blocks of σ in the i-th block π.

Theorem 79. Let $I = [\sigma, \pi]$ be an interval in Π_n, so that $|\pi| = u$ and so that there are v blocks of σ in every block of π then

$$R(I, r) = \sum_{k=1}^{r} B(v)^{u-k} u^{\underline{k}} B_{r,k} \left(R(v, 1), \ldots, R(v, r - k + 1) \right) \tag{4.101}$$

is obtained.

Proof. Firstly, set $p_i = v$ in the proof of theorem 77. Then proceed by recalling the definition of the Potential polynomials $P_k^{(u)}$

$$\begin{aligned}
\hat{R}_I(z) &= \left(\hat{R}_v(z)\right)^u \\
&= B(v)^u \left(1 + \sum_{k\geq 1} \frac{R(v,k)}{B(v)} \frac{z^k}{k!}\right)^u \\
&= B(v)^u \left(1 + \sum_{r\geq 1} P_r^{(u)} \left(\frac{R(v,1)}{B(v)}, \ldots, \frac{R(v,r)}{B(v)}\right) \frac{z^r}{r!}\right).
\end{aligned}$$

Secondly, the claim

$$\begin{aligned}
R(I,r) &= B(v)^u P_r^{(u)} \left(\frac{R(v,1)}{B(v)}, \ldots, \frac{R(v,r)}{B(v)}\right) \\
&= \sum_{k=1}^{r} B(v)^{u-k} u^{\underline{k}} B_{r,k} \left(R(v,1), \ldots, R(v, r-k+1)\right)
\end{aligned}$$

follows. □

The numbers $T(n,r)$ also obey a representation as a linear combination of Bell numbers. In order to find this linear combination a bijection analogues to Theorem 67 is used. Denote by $\mathscr{T}(n,r)$ the subset of all partitions of Π_{n+r} so that the condition 2 of Theorem 67 is satisfied, then clearly $T(n,r) = |\mathscr{T}(n,r)|$. The theorem for the linear combination in terms of Bell numbers of the $T(n,r)$ can then be stated.

Theorem 80. The numbers $T(n,r)$ obey a linear combination in terms of Bell numbers

$$T(n,r) = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} B(n+k). \tag{4.102}$$

Proof. Observe that

$$\mathscr{T}(n,r) = \bigcap_{i=1}^{r} \overline{\mathscr{C}_i}$$

holds by Theorem 67. By applying the Inclusion-Exclusion principle it is

$$\begin{aligned}
T(n,r) &= \sum_{I\subseteq\{1,\ldots,r\}} (-1)^{|I|} \left|\bigcap_{i\in I} \mathscr{C}_i\right| \\
&= \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} B(n+k),
\end{aligned}$$

as the cardinality of the subset of the partitions from Π_{n+r} with k singleton blocks of the form $\{n+i\}$ with $1 \leq i \leq r$ is $B(n+r-k)$. □

The numbers $T(n,r)$ and $R(n,r)$ are related to each other.

Theorem 81. Let n and r be non-negative integers, then the relations

$$T(n,r) = \sum_{i=0}^{r} S(r,i)R(n,i) \tag{4.103}$$

$$R(n,r) = \sum_{i=0}^{r} s(r,i)T(n,i) \tag{4.104}$$

are satisfied.

Proof. The proof of the two identities is readily established by the use of Theorem 25. Only the first of the two identities is proven here.

$$\begin{aligned}
T(n,r) &= \sum_{k=0}^{n} S(n,k)k^r \\
&= \sum_{k=0}^{n} S(n,k) \sum_{i=0}^{r} S(r,i)k^{\underline{i}} \\
&= \sum_{i=0}^{r} S(r,i) \sum_{k=0}^{n} S(n,k)k^{\underline{i}} \\
&= \sum_{i=0}^{r} S(r,i)R(n,i)
\end{aligned}$$

□

5. Conclusion

This chapter summarizes the findings made in the previous chapters and gives a brief outlook to further research questions that remained unanswered.

During the research made for this thesis it turned out that the exponential formula is an important tool that conveys simple solutions for a multitude of problems in enumerative combinatorics dealing with the partition lattice. With the help of the exponential formula and the associated Set operator it was possible to derive the exponential generating functions of the Bell numbers and the Stirling numbers of the second kind and moreover the number of chains in the partition lattice.

Utilizing the resulting exponential generating functions together with the results found in [8] it was shown in Theorem 59 that the Möbius function in the partition lattice is conveyed by the chains in the partition lattice. Moreover the interesting sum in Theorem 58 is found, that resists to have an proof using merely basic combinatorial arguments. The author tried in several attempts to find a basic combinatorial proof for Theorem 58, which could be established by a certain bijection or the involution principle, see [3, page 202ff.] for a multitude of examples utilizing the involution principle.

The tool of exponential generating functions in combination with the Isomorphism Theorem 51 allowed the author the counting of the number of chains in arbitrary intervals.

The connection between graphs and set partitions enabled the author the interpretation of set partitions in a graph theoretic sense. The Equation 4.88 and Theorem 50 showed that the use of graph theoretic viewpoints can lead to astonishing simple proofs that are related to set partitions.

The last chapter originates from the primal question for this thesis, namely the finding of amenable form of sums in the partition lattice. The author tried some serious efforts to simplify some rather simple sums, but most of the questions in this field remained unsettled and are lacking seemingly easy answers.

The sums under consideration presented in Section 4.3 could not be viewed as very strong simplifications, but merely as different representation of them, even though some benefits are given by the specification of the exponential generating function of the $R(n,r)$ in Theorem 64 that could be utilized to derive asymptotic estimates for the $R(n,r)$.

The Inclusion-Exclusion principle applied to the sums $R(n,r)$ demonstrated the intriguing combinatorial interpretations of the numbers $R(n,r)$ and $M(n,r)$. The tool of exponential generating functions could be utilized to expand the results found to arbitrary intervals in the partition lattice. Considering some special cases in Theorem 78 and Theorem 79 led to a reduction of the complexity of the sums $R(n,r)$, but a more amenable form remained infeasible to the author. The

Inclusion-Exclusion principle that led to the Theorem 34 resulted in the interesting Equation 3.42 comparing the difference of graphs with an even and odd number of edges, respectively. The author undertook great efforts to find a basic combinatorial proof for this astonishing simple equation, but all efforts remained fruitless.

Another combinatorial question that remained unsettled in this chapter is the direct combinatorial interpretation of the recurrence relation for the $R(n, r)$ presented in Theorem 61.

The number of chains in the partition lattice without repetitions that are counted by the $e_{n,k}$ defined in Theorem 53 are a starting point for a lot of question dealing with unimodularity. The author checked with the use of an computer algebra system that for all $n \geq 100$ the sequence $\{e_{n,k}\}_{k=1}^{n-1}$ is strictly unimodular, i.e. there exists a j, so that

$$e_{n,1} < \ldots < e_{n,j-1} < e_{n,j} > e_{n,j+1} > \ldots > e_{n,n-1} \tag{5.1}$$

for all $n \geq 1$. There seems to be a close connection of the unimodularity of the sequence $\{e_{n,k}\}_{k=1}^{n-1}$ and the Stirling numbers of the second kind see also [34, page 136ff.] for the unimodularity of the Stirling numbers of the second kind. The author undertook several fruitless attempts to settle the question of the unimodularity of the sequence on hand. During this survey the author was convinced, that the sequence is not merely unimodular, but also strictly logarithmic concave, i.e. the inequality

$$e_{n,k-1}e_{n,k+1} < e_{n,k}^2 \quad \text{for all } k \quad \text{with } 2 \leq k \leq n-2 \tag{5.2}$$

holds. Note that the condition of logarithmic concavity implies the unimodularity of a sequence. It is shown in [34], that the Stirling numbers of the second kind are strictly logarithmic concave. The author is convinced that the logarithmic concavity of the sequence on hand is conveyed by the logarithmic concavity of the Stirling numbers of the second kind due to Theorem 53. The conjecture of the logarithmic concavity was automatically verified for all values $n \leq 100$ by the use of a computer algebra system.

Finally the author would like to draw the readers attention to the "On-Line Encyclopedia of Integer Sequences" that is listed in the reference [28]. This database proved to be one of the most important sources during the research made for this thesis in many ways. Firstly, it helped the author to recognize already known sequences of numbers and secondly, it provided the author links to the related articles available for the sequences under consideration e.g. the alternative proof for the Möbius function in the partition lattice emerged from this database. The author was before unacquainted to the source [8] that was given for the sequence found in Table 4.2 that gave rise to the Section 3.7 and finally to Theorem 59.

A. Appendix

A.1. Notation

Symbol	Short description	First occurrence
$x \lessdot y$	y covers x	Sec. 2.1, p. 3
$[x, y]$	Interval of x and y	Sec. 2.1, p. 3
$\mathfrak{F}(x)$	Filter of x	Sec. 2.1, p. 3
$\hat{1}$	Greatest element of a poset	Sec. 2.1, p. 3
$\hat{0}$	Smallest element of a poset	Sec. 2.1, p. 3
$x \vee y$	Supremum of x and y	Sec. 2.1, p. 4
$x \wedge y$	Infimum of x and y	Sec. 2.1, p. 4
$P \times Q$	Product order of P and Q	Sec. 2.1, p. 5
$P \simeq Q$	Isomorphic posets	Sec. 2.1, p. 5
$\mathfrak{I}(P)$	Incidence algebra of the poset P	Sec. 2.3, p. 8
$f * g$	Convolution product in $\mathfrak{I}(P)$	Sec. 2.3, p. 8
ζ	Zeta-function	Sec. 2.3, p. 8
δ	Delta-function	Sec. 2.3, p. 8
μ	Möbius function	Sec. 2.3, p. 8
η	Predecessor function	Sec. 2.3, p. 10
$\Pi(X)$	The set of all set partitions of the finite set X	Sec. 3.1, p. 13
Π_n	The set of all set partitions of $\{1, \ldots, n\}$	Sec. 3.1, p. 13
$B_\pi(x)$	Block of $\pi \in \Pi(X)$ containing x	Sec. 3.1, p. 13
$x \sim_\pi y$	Equivalence relation induced by π	Sec. 3.1, p. 13
$B(n)$	Bell number	Sec. 3.1, p. 14
$S(n, k)$	Stirling numbers of the second kind	Sec. 3.1, p. 15
$S(n, 1^{k_1} \ldots n^{k_n})$	Generalized Stirling numbers of the second kind	Sec. 3.1, p. 16
$\lvert s(n, k)\rvert$	Unsigned Stirling numbers of the first kind	Sec. 3.1, p. 17
$\lvert s(n, 1^{k_1} \ldots n^{k_n})\rvert$	Generalized Stirling numbers of the first kind	Sec. 3.1, p. 17
$\mathfrak{S}_X$	The set of all permutations of the finite set X	Sec. 3.1, p. 17
$\mathfrak{S}_n$	The set of all permutations of the set $\{1, \ldots, n\}$	Sec. 3.1, p. 17
$z^{\underline{n}}$	n-th falling factorial of z	Sec. 3.1, p. 18
$z^{\overline{n}}$	n-th rising factorial of z	Sec. 3.1, p. 18
$s(n, k)$	Stirling numbers of the first kind	Sec. 3.1, p. 19
$\delta_{n,k}$	Kronecker delta	Sec. 3.1, p. 19
$G(V, E)$	Graph	Sec. 3.2, p. 20
$[V]^2$	The set of the two element subsets of V	Sec. 3.2, p. 20
K_n	The complete graph with n vertices	Sec. 3.2, p. 20
$\overline{K}_n$	The empty graph with n vertices	Sec. 3.2, p. 20
Π_G	Set of independent set partitions of G	Sec. 3.2, p. 21
$\chi(G, \lambda)$	Chromatic polynomial of G	Sec. 3.2, p. 21
$M(G, \lambda)$	Independent set polynomial of G	Sec. 3.2, p. 21
$\mathbb{C}[[z]]$	Ring of formal power series	Sec. 3.4, p. 26

$[z^n]$	Coefficient extraction operator	Sec. 3.4, p. 27
$\mathcal{G}$	Labelled class of graphs	Sec. 3.5, p. 30
$\mathcal{Z}$	Atomic labelled class	Sec. 3.5, p. 29
$\mathcal{E}$	Empty labelled class	Sec. 3.5, p. 29
$\mathcal{A} + \mathcal{B}$	Labelled sum	Sec. 3.5, p. 30
$\mathcal{A} \star \mathcal{B}$	Labelled product	Sec. 3.5, p. 31
Seq	Sequence operator	Sec. 3.5, p. 33
Set	Set operator	Sec. 3.5, p. 34
Cyc	Cycle operator	Sec. 3.5, p. 39
$B_n(z)$	Exponential polynomials	Sec. 3.5, p. 39
$\Phi_n(u)$	Polynomials of binomial type	Sec. 3.5, p. 40
$B_{n,k}(x_1, \dots, x_{n-k+1})$	Partial Bell polynomials	Sec. 3.6, p. 41
$B_n(x_1, \dots, x_n)$	Complete Bell polynomials	Sec. 3.6, p. 42
$L_n(x_1, \dots, x_n)$	Logarithmic Bell polynomials	Sec. 3.7, p. 43
$P_n^{(u)}(x_1, \dots, x_n)$	Potential polynomials	Sec. 3.7, p. 43
$f^{<\alpha>}(z)$	α-fold mapping of $f(z)$	Sec. 3.7, p. 42
$\mathbf{B}(f)$	Infinite iteration matrix of the first kind	Sec. 3.7, p. 42
$\mathbf{C}(f)$	Infinite iteration matrix of the second kind	Sec. 3.7, p. 44
$c_{n,k}(f_1, \dots, f_{n-k+1})$	First column and n-th row element of $\mathbf{C}(f)^k$	Sec. 3.7, p. 44
$\coprod_{i\in I} P_i$	Product order of the posets P_i	Sec. 4.1, p. 54
$z_{n,k}$	Number of chains between $\hat{0}$ and $\hat{1}$ in Π_n of length k with repetitions	Sec. 4.1, p. 55
$e_{n,k}$	Number of chains between $\hat{0}$ and $\hat{1}$ in Π_n of length k without repetitions	Sec. 4.1, p. 56
E_n	Number of chains between $\hat{0}$ and $\hat{1}$ in Π_n without repetitions	Sec. 4.1, p. 62
Λ	Lengyel's constant	Sec. 4.1, p. 62

A.2. Declaration of academic honesty

Hereby I, Frank Simon, assure to have written the present thesis with the title “Enumerative Combinatorics in the Partition Lattice” on my own and that I marked all the literature and resources used.

Mittweida, July 20, 2009

Place and Date | Signature

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