HOCHSCHULE MITTWEIDA
UNIVERSITY OF APPLIED SCIENCES

MASTERTHESIS

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Biconnected reliability

2015

Fakultät **Mathematik/Naturwissenschaften/Informatik**

MASTERTHESIS

Biconnected reliability

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Seminargruppe:
MA13w1-M

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Mittweida, August 2015

Bibliografische Angaben

Lange, Thomas: Biconnected reliability, 50 Seiten, 17 Abbildungen, Hochschule Mittweida (FH), Fakultät Mathematik/Naturwissenschaften/Informatik

Masterthesis, 2015

Englischer Titel

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IV. Preface

Computer communication networks play a mayor role to manage data transfer and information processing. However, the components of the network may fail - by targeted attacks or by wearout. While targeted attacks are non-random, it seems appropriate to consider wearout effects as random. Further we can assume that the components fail independently. The task of network reliability is to analyze networks in respect to the functionality of the network with consideration of wearout of its components. In most cases a network is considered as functional if a selected set of terminals can communicate. However, in practical applications additional restrictions to data capacities of the components or limited time delay in the communication apply. We will consider the special case were all connections have unit capacity while we want to transfer two data packages between selected vertices. Alternatively, one could consider the so-called biconnected reliability as the probability that the network can communicate under wearout even after a targeted attack destructed one vertex.

1 Preliminaries

1.1 Graph notations

We presume that the reader is familiar with the basics of graph theory. Therefore we will only give the used notation for common graph concepts.

In general by V we will denote the vertex set of the graph G and by E the edge set of G. $N_G(v)$ of a vertex $v \in V$ is the set of vertices adjacent to v in G and is called open neighbourhood of v.
$N_G[v] := N_G(v) \cup \{v\}$ denotes the closed neighbourhood of v.
For vertex subsets $W \subseteq V$, we define the open and closed neighbourhood by

$$N_G(W) := \bigcup_{v \in W} N_G(v) \setminus W \qquad \text{and} \qquad N_G[W] := N_G(W) \cup W.$$

δW denotes the set of edges between W and $V \setminus W$.
δ denotes the minimum degree of G and Δ is the maximum degree of G.
A separator is a vertex subset $S \subseteq V$ such that there exist two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $G = G_1 \cup G_2$, $V_1 \cap V_2 = S$. An articulation is a separator of cardinality one.
A cutset is an edge subset $C \subseteq E$, such that there exist two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $G - C = G_1 \cup G_2$ and $V_1 \cap V_2 = \emptyset$.

1.2 Graph operations

For a graph $G = (V, E)$ we use the following graph operations:

$G - e$ $:= (V, E \setminus \{e\})$ is the graph resulting from G after deletion of the edge e.
$G - v$ $:= (V \setminus \{v\}, E \setminus \delta\{v\})$ is the resulting graph from G after deletion of the vertex v and all edges incident to v.
$G - F$ $:= (V, E \setminus F)$ is the graph resulting from G after deleting all edges of F.
$G - W$ $:= (V \setminus W, \{\{x, y\} \in E | x, y \in V \setminus W\})$ is the graph resulting from G after deletion of all vertices of $W \subseteq V$ and all edges incident to vertices of W.
$G + e$ $:= (V, E \cup \{e\})$ is the graph resulting from G after insertion of the edge e (resulting multiple edges will be conserved).
$G + v$ $:= (V \cup \{v\}, E)$ is the graph after insertion of a new vertex v.
$G{<}W{>}$ $:= (W, \{x, y\} \in E | x, y \in W)$ is the induced subgraph of G.
$G(F)$ $:= (V, F) = G - (E \setminus F)$.

G/e is the resulting graph of G after contraction of the edge $e \in E$ (identifying the end vertices of e and conserving multiple edges).

$G+H$ $:= (V(H) \cup V(G), E(H) \cup E(G) \cup \{\{u,v\} | u \in V(H), v \in V(G)\})$ is the join of the graphs G and H. We will assume that $V(H) \cap V(G) = \emptyset$.

For a probabilistic graph (see Definition 2.1), we have all those graph operations, while leaving the edge probabilities unchanged (for $G+e$ and $G+H$ the probability of the new edge(s) will be given explicitly). Additionally, we have the operation $G|_{p_e=k}$, which will leave the graph unchanged while changing the edge probability of e to the value k.

1.3 Graph connectivity

Since the focus of this thesis is the reliability of two-connected graphs, we will define connectivity in detail and characterize two-connected graphs.
The local vertex-connectivity function $\kappa(x,y)$, defined for every pair of non-adjacent vertices, is the minimum number of vertices, whose omission from G disconnects x and y. The local edge-connectivity function $\lambda(x,y)$, defined for every pair of vertices, is the minimum number of edges, whose omission from G disconnects x and y. The vertex-connectivity $\kappa(G) = \min \kappa(x,y)$ is the global minimum of the local vertex-connectivity. Similarly, the edge-connectivity $\lambda(G) = \min \lambda(x,y)$ is the global minimum of the local edge-connectivity. For the complete graph of order n, K_n, we define the connectivity as: $\kappa(K_n) = \lambda(K_n) = n-1$. A graph G is called k-connected if and only if $\kappa(G) \geq k$ and is called k-edge-connected if and only if $\lambda(G) \geq k$.
In this thesis we will use the terms two-connected, biconnected and non-separable as synonyms and therefore explicitly exclude the K_2 from the latter. A block of a graph is a biconnected component of a graph.
A vertex set V_1 is connected to a vertex v via a vertex set V_2 in G if and only if the following holds:

- V_1,V_2 and v are in the same connected component of G
- In the graph $G - V_2$ for all $u \in V_1$ holds: u and v are not connected.

The most important result linking connectivity to pathsets is due to Menger's famous theorem [Men27] which we will now state as presented by Bollobás [Bol04].

Theorem 1.1 (Menger's Theorem) *A graph G is k-connected (resp. k-edge-connected) if and only if for any two vertices there are k disjoint (resp. k edge-disjoint) paths joining them.*

Proof: For the proof see [Men27]. Other elegant proofs dued to Dirac and Pym can be found in [Dir66] and [Pym69]. □

Therefore we call two vertices u, v biconnected if and only if there exist two disjoint paths between u and v, that means u and v are in a common block.

Two-connected graphs can be characterized by the following theorem:

Theorem 1.2 *Let G be a graph with $|V| \geq 3$. Then the following conditions are equivalent.*

- *G is two-connected.*
- *G has no articulation.*
- *Given any two vertices there is a cycle containing them.*
- *Given any vertex and any edge there is a cycle containing them.*
- *Given any two edges there is a cycle containing them.*

Proof: The proof can be found in [Plu68]. □

1.4 Special graph classes

For the complete bipartite graph $K_{a,b} = (A \cup B, \{\{x,y\}, x \in A, y \in B\})$, $a = |A|, b = |B|$ we will denote vertices $v \in A$ as a-vertices and vertices $v \in B$ as b-vertices (see Figure 1.1). We can extend this terminology to all bipartite graphs. Further without loss of generality, for the complete bipartite graph, we will assume $a \leq b$.

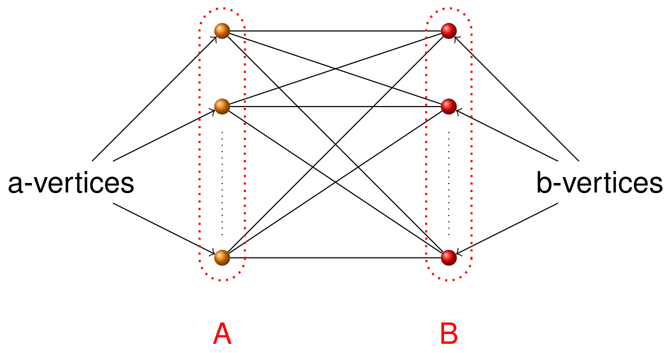

Figure 1.1: Complete bipartite graph with a- and b-vertices

A two-tree is a graph which can be generated by the following procedure:

1. Start with $G = K_3$.
2. Stop or go to step 3.
3. Select an edge $e = \{u, w\} \in E$.
4. Add a new vertex v and the edges $\{u, v\}$ and $\{v, w\}$ to G.
5. Go to step 2.

A simple two-tree is a graph, where no edge is selected more then once in the procedure. A two-path is a simple two-tree where only edges are selected with $\min\{\deg u, \deg w\} = 2$. We introduce the following notations:

- P_n^2 is an arbitrary two-path with $|V| = n$.
- T_n^2 is an arbitrary simple two-tree with $|V| = n$.
- $T_{n,k}^2$ is an arbitrary two-tree with $|V| = n$ where k counts the number of iterations where an edge was selected in step 3 which was already selected in a previous iteration of the procedure.

Remark 1.3 The value of k in $T_{n,k}^2$ is independent of the order in which the edges are selected.

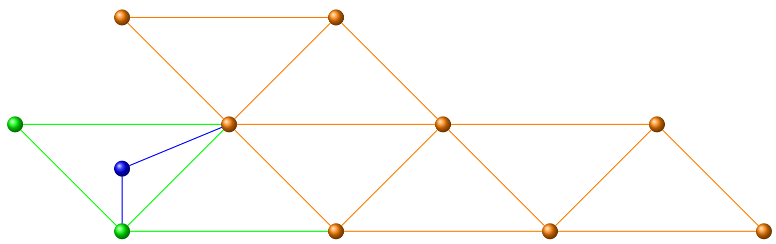

Figure 1.2: Example of a two-path P_8^2(orange), a simple two-tree T_{10}^2 (orange+green) and a two-tree $T_{11,1}^2$(orange+green+blue)

Remark 1.4 For every simple two-tree there exists a planar embedding, such that all vertices are on the outside. For a two-tree, which is not simple, no such embedding exists.

The wheel graph W_n is the resulting graph of the join $C_n + K_1$.

1.5 Partitions and compositions

Let π be a partition of the set $\{1,\ldots,n\}$. The type of π, denoted by $\lambda(\pi)$, is an integer partition of n that gives the block size distribution of π. For a given integer partition $\lambda = (\lambda_1,\ldots,\lambda_j)$ of n, we denote by $k_i(\lambda)$ the number of parts of λ that are equal to i, $i = 1,\ldots,n$ and define

$$k_\lambda = (k_1(\lambda),\ldots,k_n(\lambda)).$$

We write $\lambda \vdash n$ whenever λ is a partition of n and denote the number of parts of λ by $|\lambda|$. Further we write $\lambda \vdash (n,b)$ if λ is a partition of n with $k_i(\lambda) = 0$ for all $i < b$, that means that each block of π has at least size b. We use the following notations:

$$\binom{n}{\lambda} := \binom{n}{\lambda_1,\lambda_2,\ldots,\lambda_{|\lambda|}}$$

$$k_\lambda! := \prod_{i=1}^{n} k_i(\lambda)!$$

$$\left\{ \begin{matrix} n \\ \lambda \end{matrix} \right\} := \frac{1}{k_\lambda!} \binom{n}{\lambda}$$

where the last expression equals the number of set partitions of $\{1,\ldots,n\}$ of the given type $\lambda \vdash n$.
We write $\lambda \vDash n$ if $\lambda = (\lambda_1,\ldots,\lambda_r)$ is a composition (ordered integer partition) of n. We introduce the following additional notations:

- $\lambda \vDash (n,s) :\Leftrightarrow \lambda \vDash n$ and $|\lambda| = s$
- $\lambda \vDash (n,s,b) :\Leftrightarrow n = \sum_{i=1}^{s} \lambda_i$ and $\lambda_i \geq b$ for all i
- $\binom{n}{\lambda} := \binom{n}{\lambda_1,\ldots,\lambda_{|\lambda|}}$

Remark 1.5 It can easily be verified, that $\lambda \vDash (n,s,b) \Leftrightarrow \lambda - (b-1) \vDash (n-(b-1)\cdot s, s)$ where $\lambda - (b-1)$ means that each part of λ is reduced by $b-1$. Therefore the set $\{\lambda : \lambda \vDash (n,s,b)\}$ for given n, s and b is finite even for $b \leq 0$.

2 Reliability measures

2.1 Probabilistic graphs and biconnected reliability

Definition 2.1 A probabilistic graph $G = (V,E)$ is an undirected graph together with a mapping $p : E \to [0,1]$. $p_e := p(e)$ is the probability of e being operating and $q_e := 1 - p_e$ is the failure probability of $e \in E$.
A state of the graph is characterized by the set of operating edges $F \subseteq E$.

For the remainder of this thesis, the term graph will refer to a probabilistic graph unless stated otherwise.
The following reliability measures have been extensively studied in a vast amount of previous papers: The K-terminal reliability $R_K(G)$ is the probability that all vertices of a vertex subset K of a probabilistic graph G are connected. The two-terminal reliability $R_{st}(G)$ is the special case with $K = \{s,t\}$ and the all-terminal reliability $R(G)$ has $K = V$. We will use these to define similar reliability measures for higher connectivity restraints.

Definition 2.2 The K-terminal biconnected reliability $R_K^2(G)$ is the probability that all vertices of a vertex subset K of a probabilistic graph G are in the same block.

Definition 2.3 The two-terminal biconnected reliability $R_{st}^2(G)$ is the probability that the vertices s and t of a probabilistic graph G are in the same block.

Definition 2.4 The (all-terminal) biconnected reliability $R^2(G)$ is the probability that a probabilistic graph G is biconnected.

Remark 2.5 To avoid special case destinction, the complete graph with two vertices, K_2, will not be considered as a block/biconnected.

The reliability can be described by suitable pathsets which we will now define more formally.

Definition 2.6 For given $K \subseteq V$ and a probabilistic graph $G = (V,E)$ a success set $F \subseteq E$ is an edge subset such that all vertices of K are in the same block of (V,F). A success set F is called minimal, if no proper subset of F is a success set.

Definition 2.7 For given $K \subseteq V$ and a probabilistic graph $G = (V,E)$ a failure set $C \subseteq E$ is an edge subset such that $E \setminus C$ is not a success set of G and K. A failure set C is called minimal, if no proper subset of C is a failure set.

With $\mathscr{F}$ and $\mathscr{C}$ we will denote the set of success sets and failure sets of G and K. $\mathscr{F}'$ and $\mathscr{C}'$ will denote the corresponding sets of minimal success and failure sets.

Definition 2.8 Let G be a two-connected graph.
An edge e is essential, if and only if e is part of every minimal success set.
An edge e is irrelevant, if and only if e is part of no minimal success set.

Since success and failure sets correspond to states of the probabilistic graph and edges fail independently, we can calculate the K-terminal-biconnected reliability of G, where we compound the edge probabilities in the vector **p**, by the following formulas:

$$R_K^2(G,\mathbf{p}) = \sum_{F\in\mathscr{F}} \prod_{e\in F} p_e \prod_{f\in E\setminus F} (1-p_f) \tag{2.1}$$

$$=1-\sum_{C\in\mathscr{C}} \prod_{e\in C}(1-p_e) \prod_{f\in E\setminus C} p_f \tag{2.2}$$

where from now on for simplicity we will omit **p** and simply write $R_K^2(G)$.

Remark 2.9 Since every superset of a minimal success/failure set is a success/failure set as well, $\mathscr{F}$ and $\mathscr{C}$ are completly described by $\mathscr{F}'$ and $\mathscr{C}'$. Therefore, the biconnected-reliability can be calculated via the minimal success and failure sets, even though an explicit formula will not be given here.

Remark 2.10 Let G be a biconnected graph. If e is an essential edge, $G-e$ is not biconnected and hence it holds $R_K^2(G-e)=0$. If e is an irrelevant edge, it holds $R_K^2(G)=R_K^2(G-e)$. Further, $G-e$ has the same minimal success and failure sets as G. Hence, the removal of an irrelevant edge e does not change the characterisation of other edges as essential and irrelevant edges.

If all edges fail with the same probability p, the biconnected reliability becomes a polynomial in p or $q=1-p$:

$$R_K^2(G,p) = \sum_{F\in\mathscr{F}} p^{|F|}(1-p)^{|E|-|F|} \tag{2.3}$$

$$=1-\sum_{C\in\mathscr{C}} (1-p)^{|C|}p^{|E|-|C|} \tag{2.4}$$

Non-isomorphic graphs (even with unequal number of edges) may have the same biconnected reliability polynomial, see Figure 2.1.

There are several representations of the biconnected-reliability polynomial. In every case we could as well sum from 0 to m, however for simplicity we will omit coefficients

Figure 2.1: Two non-isomorphic graphs which have the same biconnected reliability polynomial $R^2(G_1, p) = R^2(G_2, p) = p^4$.

which will always be zero. The success form of the biconnected reliability is given by

$$R^2(G,p) = \sum_{i=n}^{m} n_i p^i (1-p)^{m-i} \tag{2.5}$$

where n_i is the number of success sets (spanning 2-connected subgraphs) of G of cardinality i. A two-vertex-connected subgraph with exactly $m = n$ edges can only be a Hamilton cycle and hence n_n is the number of Hamilton cycles of G and we get the following theorem:

Theorem 2.11 *The calculation of the biconnected reliability is NP-hard.*

Proof: Since n_n is the number of Hamilton cycles of G and the decision problem whether a given graph contains a Hamilton cycle is NP-complete [Kar72], the calculation of the coefficients of $R^2(G,p)$ is NP-hard. Assume we can calculate $R^2(G,\mathbf{p})$ for arbitrary values of **p**. By calculating $R^2(G,p)$ for m different values p_i with $0 < p_1 < p_2 < \ldots < p_m < 1$ we derive a system of m independent linear equations. Hence, via Gaussian elimination, we could reconstruct the coefficients of the polynomial in polynomial time. Therefore the calculation of $R^2(G,\mathbf{p})$ must be NP-hard. □

By expanding the success form in terms of p we can get the simple form:

$$R^2(G,p) = \sum_{i=n}^{m} a_i p^i \tag{2.6}$$

where a combinatorical interpretation is not yet known unless $i = n$, where $a_n = n_n$ holds. By representing $R^2(G,p)$ in terms of failure sets, we get the cut-form:

$$R^2(G,p) = \sum_{i=\lambda(G)-1}^{m} c_i (1-p)^i p^{m-i} \tag{2.7}$$

where c_i is the number of failure sets of cardinality i.
The following relations can be used to transform the success and failure forms into each other:
It holds

$$n_i + c_{m-i} = \binom{m}{i}, \tag{2.8}$$

since for every edge subset $F \subseteq E$, $|F| = i$ one of the following things can happen: F is a success set (counted by n_i) or it is not, in which case $E \setminus F$ must be a failure set (counted by c_{m-i}).
The simple form is linked to the success form via the following relation:

$$n_l = \sum_{i=0}^{l-1} \binom{m-l+i}{i} a_{l-i} \qquad (2.9)$$

which follows immediately from the expansion of the success form to derive the simple form. Further representations of the reliability polynomial which can be transferred to the biconnected reliability can be found for example in [Col91].
Since we can calculate the biconnected reliability via pathsets, we get the following formula:

Theorem 2.12 *Let G be a probabilistic graph. Then for every edge $e \in E$ and every $K \subseteq V$ holds:*

$$R_K^2(G) = (1-p_e) \cdot R_K^2(G-e) + p_e \cdot R_K^2(G|_{p_e=1})$$

Proof: The K-terminal biconnected reliability can be expressed via pathsets:

$$\begin{aligned}
R_K^2(G) = & \sum_{F \in \mathscr{F}(G)} \prod_{f \in F} p_f \prod_{g \in E \setminus F} (1-p_g) \\
= & \sum_{F \in \mathscr{F}(G), e \in F} \prod_{f \in F} p_f \prod_{g \in E \setminus F} (1-p_g) + \sum_{F \in \mathscr{F}(G), e \notin F} \prod_{f \in F} p_f \prod_{g \in E \setminus F} (1-p_g) \\
= & p_e \cdot \sum_{F \in \mathscr{F}(G), e \in F} \prod_{f \in F, f \neq e} p_f \prod_{g \in E \setminus F} (1-p_g) \\
& + (1-p_e) \cdot \sum_{F \in \mathscr{F}(G), e \notin F} \prod_{f \in F} p_f \prod_{g \in E \setminus F, g \neq e} (1-p_g) \\
= & p_e \cdot \sum_{F \in \mathscr{F}(G|_{p_e=1}), e \in F} \prod_{f \in F} p_f \prod_{g \in E \setminus F} (1-p_g) \\
& + (1-p_e) \cdot \sum_{F \in \mathscr{F}(G-e)} \prod_{f \in F} p_f \prod_{g \in (E-\{e\}) \setminus F} (1-p_g) \\
= & p_e \cdot R_K^2(G|_{p_e=1}) + (1-p_e) \cdot R_K^2(G-e)
\end{aligned}$$

□

Remark 2.13 Unlike for the all-terminal reliability, in general it does not hold

$$R_K^2(G|_{p_e=1}) = R_K^2(G/e),$$

see for example Figure 2.2.

Figure 2.2: Contracting the edge a changes the biconnected reliability (unless $p_b = 1$ or $p_e = 0$). We have $R^2(G|_{p_a=1}) = p_b p_c p_d$ and $R^2(G/a) = (p_b + p_e - p_b p_e) p_c p_d$.

2.2 Other reliability measures

While the focus of this work is on the biconnected reliability, some results will be extended to the following reliability measures:

Definition 2.14 The K-terminal k-connected reliability $R_K^k(G)$ is the probability that all vertices of a vertex subset K of a probabilistic graph G are in the same k-connected component of G.

Definition 2.15 The (all-terminal) k-connected reliability $R^k(G)$ is the probability that a probabilistic graph G is k-connected.

Definition 2.16 The two-edge connected reliability $R_{2-ec}(G)$ is the probability that a probabilistic graph G is two-edge connected.

Definition 2.17 The k-edge connected reliability $R_{k-ec}(G)$ is the probability that a probabilistic graph G is k-edge connected.

For the corresponding reliability polynomials we can again derive the different representations in a similar fashion. For the two-edge connected reliability and k-edge connected reliability those can be found in [Rei15].

3 Reductions

3.1 Series-Parallel reductions and articulations

Theorem 3.1 *Let $e \in E$ be a loop of G. Then for all $K \subseteq V$ it holds:*

$$R_K^2(G) = R_K^2(G-e).$$

Proof: The K-terminal biconnected reliability can be calculated via the set of all inclusion-minimal success sets. Assume e is part of a minimal success set F, that means that all vertices of K are within the same block in (G,F) but not in $(G,F-e)$. But since e is a loop, (G,F) and $(G,F-e)$ have exactly the same blocks, therefore e can not be a part of a minimal success set and the theorem holds. □

Theorem 3.2 (Parallel reduction) *Let $F := \{f_1, \dots, f_k\}$ be a set of $k \geq 2$ parallel edges incident to the vertices u and v with the corresponding probabilities of failure $q_1, \dots q_k$. Let $e = \{u,v\}$ be a newly introduced edge between u and v with failure probability $q_e = \prod_{i=1}^{k} q_i$.*
Then for all $K \subseteq V, K \neq \{u,v\}$, the following statement holds:

$$R_K^2(G) = R_K^2(G-F+e).$$

For $K = \{u,v\}$ it holds:

$$R_{uv}^2(G) = 1 + (R_{uv}^2(G-F) - 1) \cdot \prod_{i=1}^{k} q_i + (R_{uv}(G-F) - 1) \cdot \prod_{i=1}^{k} q_i \cdot \sum_{f \in F} \frac{p_f}{q_f}.$$

Proof: Every minimal failure set of the altered graph which does not use e is exactly a minimal failure set in the original graph. So now consider minimal failure sets of the new graph which use e. In the original graph those sets without e form a failure set with all edges of $e_1, \dots, e_k$. On the other hand, no minimal failure set uses $e_1, \dots, e_k$ partly, and therefore the minimal failure sets of the original graph and the altered graph have an one-to-one correspondence with the given new probability to account for the failure of all edges $e_1, \dots, e_k$.
The result for $K = \{u,v\}$ follows immediately from Theorem 2.12. □

Corollary 3.3 *For the biconnected reliability it is sufficient to consider simple graphs unless we consider the biconnected reliability polynomial.*

Theorem 3.4 (Articulations) *Let $G = (V,E)$ be a graph and $v \in V$ an articulation of G.*

Then for all $K \subseteq V$ holds:

$$R_K^2(G) = \begin{cases} R_K^2(G_1), & \textit{if } K \subseteq V_1 \\ R_K^2(G_2), & \textit{if } K \subseteq V_2 \\ 0, & \textit{otherwise.} \end{cases}$$

Proof: If $K \not\subseteq V_1$ and $K \not\subseteq V_2$, then K contains vertices v_1, v_2 such that $v_1 \in V_1 \setminus \{v\}$, $v_2 \in V_2 \setminus \{v\}$. Since v is an articulation in G, all paths between v_1 and v_2 have v in common, therefore the biconnected reliability is 0. Without loss of generality now let $K \subseteq V_1$. Then no inclusion-minimal success set of G contains edges of E_2 since these edges would form a loop or circle from v to v. Hence, all edges of E_2 are irrelevant and we get $R_K^2(G) = R_K^2(G - E_2) = R_K^2(G_1)$ (isolated vertices in $V \setminus K$ can be omitted). □

Theorem 3.5 (Series-Reduction) *Let $v \in V$ be a vertex of G with $\deg v = 2$. Let e_1, $e_2 \in E$ be the edges incident to v with the corresponding probabilities p_1 and p_2. Furthermore, let $u, w \in N_G(v)$ be the vertices adjacent to v and $e := \{u, w\}$ be a new edge incident to u and w. Then the following holds:*

- *If $v \notin K$: $R_K^2(G) = R_K^2(G - v + e)$ with $p_e = p_1 p_2$*
- *If $v \in K$:*

$$\begin{aligned} R_K^2(G) = & \quad p_1 p_2 \cdot R_K^2(G|_{p_1 = p_2 = 1}) \\ = & \quad p_1 p_2 \cdot R_{K \cup \{u,w\}}^2(G - v + e|_{p_e = 1}). \end{aligned}$$

Proof: First, let $v \notin K$. If a minimal success set in $G - v + e$ does not contain e, it is a minimal success set in G as well. Further, there exists a one-to-one corresponding of minimal success sets of $G - v + e$ containing e and minimal success sets of G containing e_1 and e_2 instead.
Now, let $v \in K$. If e_1 or e_2 fail, then v is no longer in a block with the other vertices of v. Therefore, by Theorem 2.12 the first equality holds. For K to be in the same block in G there must be vertex-disjoint paths from every vertex of K to u and w. If we introduce the new edge e between u and w, then K, u and w are in the same block in $G - v + e$. □

Corollary 3.6 *The biconnected reliability of series-parallel graphs can be calculated in linear time.*

3.2 Reductions on separators of cardinality two

Let $\{c_1, c_2\}$ be a separator of cardinality two of $G = (V, E)$. Let $e \in E_1 \cap E_2$, if existing, denote the edge between the two cut-vertices.

Theorem 3.7 (Two-terminal biconnected reliability) *Let v be a new vertex and $e_1 = \{c_1, v\}$, $e_2 = \{c_2, v\}$ two edges incident to v and the cut vertices with $p_{e_1} = p_{e_2} = 1$.*

Let $s,t \in V$ *such that* $\{s,t\} \neq \{c_1,c_2\}$. *Let* $f = \{c_1,c_2\}$ *be a new edge with* $p_f := R_{c_1c_2}(G_2 - e)$. *For the two-terminal reliability* $R^2_{st}(G)$ *holds:*

- *If* $\{s,t\} \subseteq V_1$*:* $R^2_{st}(G) = R^2_{st}(G_1 + f)$*,*
  analogously for $\{s,t\} \subseteq V_2$ *with adapted* p_f*.*
- *If* $s \in V_1 \setminus V_2$ *and* $t \in V_2 \setminus V_1$*:*
  $R^2_{st}(G) = R^2_{sv}(G_1 + v + e_1 + e_2 - e) \cdot R^2_{vt}(G_2 + v + e_1 + e_2 - e)$.

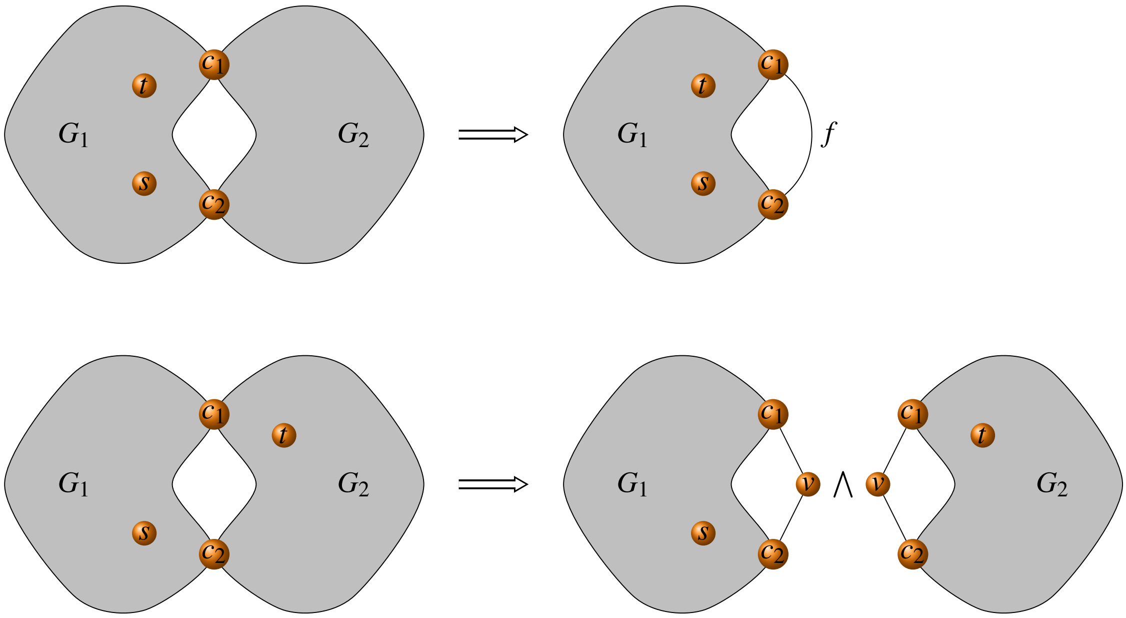

Figure 3.1: Graph splitting on separating vertex set of cardinality two to calculate $R^2_{st}(G)$ with $p_f := R_{c_1c_2}(G_2 - e)$

Proof: Let $\{s,t\} \subseteq V_1, \{s,t\} \neq \{c_1,c_2\}$. Consider the minimal success sets in G. If such a set contains no edge of E_2, it is a minimal success set of G_1. All other minimal success sets consist of two vertex-disjoint paths, one reaching t over c_1, some vertices of G_2 and c_2 and the other path directly. If we now replace the path from c_1 over some vertices of G_2 to c_2 by the edge f, we get a minimum success set of $G_1 + f$, where the probability for this newly introduced edge f is the probability that c_1 and c_2 are connected in $G_2 - e$. Now consider that s and t are on different sides of the cut. Then every success set must contain vertex-disjoint paths from s over c_1 to t and from s over c_2 to t. The paths from s to c_1 and c_2 are exclusively in G_1 while the paths from c_1 and c_2 to t are exclusively in G_2. Since the edges fail independently, we can calculate the probabilities for these subpaths independently. The probability that two vertex-disjoint paths between s and c_1 and c_2 exist, is the chance that s and a new vertex v connected solely to c_1 and c_2 with non-failing edges is in a block with s. Therefore, we get $R^2_{s,v}(G_1 + v + e_1 + e_2)$. Since the two disjoint paths between s and t must contain c_1 and c_2 respectively, no minimal success set contains e and therefore this edge is irrelevant. □

Theorem 3.8 (Two-terminal biconnected reliability) *Let* $\{s,t\} = \{c_1, c_2\}$ *and* $\{s,t\} \notin E$. *Then*

$$R^2_{st}(G) = R^2_{st}(G_1) + R^2_{st}(G_2) - R^2_{st}(G_1) \cdot R^2_{st}(G_2) + \left(R_{st}(G_1) - R^2_{st}(G_1)\right) \cdot \left(R_{st}(G_2) - R^2_{st}(G_2)\right).$$

Proof: The following events result in s and t being biconnected:

- There exist two vertex-disjoint paths between s and t in G_1.
- There exist two vertex-disjoint paths between s and t in G_2.
- G_1 and G_2 each contain one path from s to t.

The probability for exactly one path in G_1 can be described as

$$\left(R_{st}(G_1) - R^2_{st}(G_1)\right).$$

The theorem follows by inclusion-exclusion. □

Theorem 3.9 (K-terminal biconnected reliability) *Let* $f = \{c_1, c_2\}$ *be a new edge with* $p_f = R_{c_1c_2}(G_2 - e)$ *and* v *a new vertex as in Theorem 3.7. For all* $K \subseteq V$ *with* $|K| \geq 3$ *holds:*

- *If* $K \subseteq V_1$*:* $R^2_K(G) = R^2_K(G_1 + f)$.
  Analogously for $K \subseteq V_2$ *with adapted* p_f.
- *If* $K \nsubseteq V_1$ *and* $K \nsubseteq V_2$*:*

$$\begin{aligned} R^2_K(G) &= R^2_{K\cup\{c_1,c_2\}}(G-e) \\ &= R^2_{K'_1}(G_1 + v + e_1 + e_2 - e) \cdot R^2_{K'_2}(G_2 + v + e_1 + e_2 - e) \end{aligned}$$

with $(K \cap V_i) \cup \{v\} \subseteq K'_i \subseteq (K \cap V_i) \cup \{v, c_1, c_2\}$, $i \in \{1,2\}$

Remark 3.10 The second case of the theorem allows us to arbitrarily select/deselect the vertices c_1 and c_2 for the terminal set K which in some cases may allow us to reach the special cases of two-terminal or all-terminal biconnected-reliability.

Proof: First, consider the case that $K \subseteq V_1$. For every minimal success set of G the following holds: Either the success sets contains no edge of $E_2 \setminus \{e\}$ or it contains exactly one path connecting c_1 and c_2 in $G_2 - e$. The former corresponds to minimal success sets of G_1, the latter corresponds to minimal success sets of $G_1 + f$ containing f.
Now consider the case that K contains vertices of both V_1 and V_2. Let $K_1 := K \setminus V_2 \neq \emptyset$ and let $K_2 = K \setminus V_1 \neq \emptyset$. For all vertices of K to build a block in the remaining graph, every success set must contain two vertex-disjoint paths between every vertex of K_1 and K_2 which corresponds to vertex-disjoint paths of every vertex of K_1 to c_1 and c_2 in G_1 and

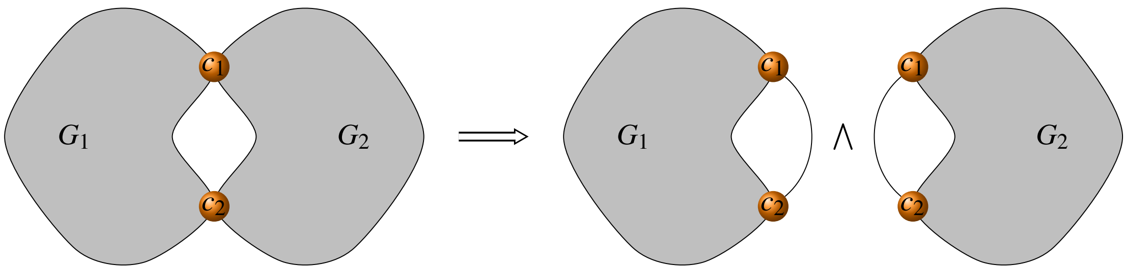

Figure 3.2: Graph splitting on separating vertex set of cardinality two to calculate $R^2(G)$.

vertex-disjoint paths of every vertex of K_2 to c_1 and c_2 in G_2. Introducing the new vertex v with incident non-failing edges e_1 and e_2 this corresponds to $R^2_{K_i}(G_i+v+e_1+e_2)$ with $i \in \{1,2\}$, respectively. Since the two vertex-disjoint paths of every vertex of K_1 and K_2 always contain the vertices c_1 and c_2, c_1 and c_2 lie in the same block as K for every success set. Hence, we can (partly) include c_1 and c_2 into K without changing the success sets and thus the probability. Since the vertex-disjoint paths contain c_1 and c_2 respectively, e is not in any minimal success set and thus irrelevant and the theorem follows. □

Theorem 3.11 (All-terminal biconnected reliability) *Let $f = \{c_1, c_2\}$ be a newly introduced edge with $p_f = 1$. For the all-terminal biconnected reliability $R^2(G)$ holds:*

$$R^2(G) = R^2(G_1+f) \cdot R^2(G_2+f)$$

Proof: The theorem follows immediately from Theorem 3.9 and Theorem 3.5 for $K = V$. □

3.3 Extension to higher connectivity demands

Theorem 3.12 *Let e be a loop in G with $|V| \geq 2$. Then for arbitrary k, the K-terminal k-connected reliability $R^k_K(G)$ and the k-edge connected reliability $R_{k-ec}(G)$ fulfill:*

$$R^k_K(G) = R^k_K(G-e) \quad \text{and}$$
$$R_{k-ec}(G) = R_{k-ec}(G-e).$$

Proof: Both reliability measures can be expressed by means of their minimal success sets. A loop does not influence connectivity, hence it is in no minimal success set and in consequence irrelevant. □

Theorem 3.13 (Parallel edges) *Let $F = \{f_1, \ldots, f_j\}$ be a set of j parallel edges incident to the vertices $u, v \in V$. Let $e = \{u, v\}$ be a newly introduced edge with failure probability*

$q_e := \prod_{i=1}^{j} q_{f_i}$. *Then*

$$R_K^k(G) = R_K^k(G - F + e).$$

Proof: Consider minimal failure sets of the altered graph $G - F + e$. Every minimal failure set which does not contain e is also a failure set of the original graph. So now consider minimal failure sets containing e. These sets are minimal failure sets of G after replacing e with all edges of F. In the cutform of $R_K^k(G)$ these sets are accounted by the term $\prod_{i=i}^{j} q_{f_i}$. □

Corollary 3.14 *For the K-terminal k-connected reliability it is sufficient to consider simple graphs unless we consider the corresponding polynomial.*

Theorem 3.15 (Articulations) *Let v be an articulation in G. Then for the k-edge connected reliability and the K-terminal k-connected reliability holds:*

$$R_{k-ec}(G) = R_{k-ec}(G_1) \cdot R_{k-ec}(G_2) \qquad \text{and}$$

$$R_K^k(G) = \begin{cases} R_K^k(G_1), & \text{if } K \subseteq V_1 \\ R_K^k(G_2), & \text{if } K \subseteq V_2 \\ 0, & \text{else.} \end{cases}$$

Proof: Due to Menger's theorem, for the state of G to be k-edge connected, for each pair of vertices $\{u,w\}$, there need to be at least k edge- disjoint paths connecting these vertices. Consider $\{u,w\}$ to be in the same subgraph, say G_1. Then the k paths will not use any edges of E_2 since they would form a loop or circle from v to v. Therefore those vertices are in the same k-edge connected component of G, if and only if they are in the same k-edge connected component of G_1. The same is true if $\{u,w\} \subseteq V_2$. Since the corresponding edge sets are independent, we get $R_{k-ec}(G) = R_{k-ec}(G_1) \cdot R_{k-ec}(G_2)$. It remains to show that only if G_1 and G_2 are k-edge connected all pairs of vertices $\{u,w\}$, such that $u \in V_1 \setminus \{v\}$ and $w \in V_2 \setminus \{v\}$, are in the same k-edge connected component. Due to Menger's theorem for $\{u,w\}$ to be in the same connected component, there need to be k edge-disjoint intact paths between u and w. Since v is a cutvertex, all those k paths contain v and the paths between u and v and v and w can be considered independently of each other. Hence, for all pairs of vertices $\{u,w\}$ in the different subgraphs to be k-edge connected, there need to be k edge-disjoint paths from every vertex u to v and from every vertex w to v. This corresponds to G_1 and G_2 being k-edge connected.
For the K-terminal k-connected reliability all vertices of K must be in the same k-connected component. If K contains vertices of both V_1 and V_2, these would have a local vertex-connectivity of one and therefore $R_K^k(G) = 0$. If all vertices are in the same block, say G_1, no minimal success set contains edges of E_2. □

Theorem 3.16 (Separator of cardinality at most k) *Let $U \subseteq V$ be a separator in G. Let*

$K_1 := K \cap V_1 \setminus U \neq \emptyset$ and $K_2 := K \cap V_2 \setminus U \neq \emptyset$. Furthermore, let $F = \{\{x,y\}|x,y \in U\}$ with $p_f := 1$ for all $f \in F$.
Then for $|U| < k$, it holds:

$$R_K^k(G) = 0.$$

For $|U| = k$, it holds:

$$R_K^k(G) = R_{K_1 \cup U}^k(G_1 + F) \cdot R_{K_2 \cup U}^k(G_2 + F).$$

Proof: If $|U| < k$, for $x \in K_1, y \in K_2$ clearly holds: $\kappa(x,y) \leq |U| < k$ and therefore x and y can not be in a k-connected component after edge failure.
So now assume $|U| = k$. Due to Menger's theorem for all vertices $\{x,y\} \subseteq K$ there need to be k disjoint paths after edge failure.
We will show the following:

- There exist k disjoint paths between all vertices of K, if and only if for every vertex of K, there exist disjoint paths to every vertex of U.
- There exist disjoint paths to all vertices of U for every vertex of K_1, if and only if the vertices of K_1 and U are in the same k-connected component in $G_1 + F$

Since the edge sets E_1 and E_2 are independent, those two results combined yield the theorem.
First, assume that there exists a vertex $x \in K_1$ which does not have disjoint paths to all vertices of U. Pick an arbitrary vertex $y \in K_2$. Since U is a separator of K_1 and K_2, all paths between x and y have to traverse through U. Since x does not have disjoint paths to all vertices of U and $|U| = k$, there can not be k disjoint paths between x and y.
Now assume that for every vertex of K there exist disjoint paths to every vertex of U. First, consider the case that x and y are in different subgraphs. Then the disjoints paths from x to U and y to U are disjoint and therefore by combining the paths reaching the same vertex of U, there exist k disjoint paths between x and y. By that construction follows, that every vertex $x \in K_1$ and $y \in K_2$ are in the same k-connected component (containing U as well). Hence, two vertices $x_1, x_2 \in K_1$ must be in the same k-connected component as well, since different k-connected components have an intersect of at most $k - 1 < |U|$.
Now we will show, that exactly then K_1 and U are in the same k-connected component in $G_1 + F$. First assume, there exists a vertex $x \in K_1$ which does not have disjoint paths to all vertices of U. Let $W \subseteq U$ be an inclusion-minimal subset such that there do not exist disjoint paths to all vertices of W which do not use $U \setminus W$. Then there clearly exists a separator, denoted by C between x and W of cardinality at most $|W| - 1$. Therefore $C \cup (U \setminus W)$ is a separator of cardinality at most $|U| - 1 = k - 1$ between x and W.
Now again assume that for each vertex $x \in K_1$ there exist disjoint paths to all vertices of U. We already showed that all vertices of K_1 are in the same k-connected component in G_1. Now for every vertex $u \in U$ the disjoint paths of x to all vertices of U when

prolonged by edges $\{w,u\}$ for each $w \in U \setminus \{u\}$ form k disjoint paths between x and u and therefore they are in the same k-connected component in $G_1 + F$. Since (U,F) is the complete graph K_k, all vertices of U are in the same k-connected component of $G_1 + F$ by default. □

Remark 3.17 Observe that $R^k_K(G) \leq R^k_{K_1 \cup U}(G_1 + F) \cdot R^k_{K_2 \cup U}(G_2 + F)$ holds in general, since every success set of G corresponds to success sets in $G_1 + F$ and $G_2 + F$. However, equality in general does not hold for $|U| > k$. An edge set which results in a failure state may become a valid success set after splitting, see Figure 3.3 for an example.

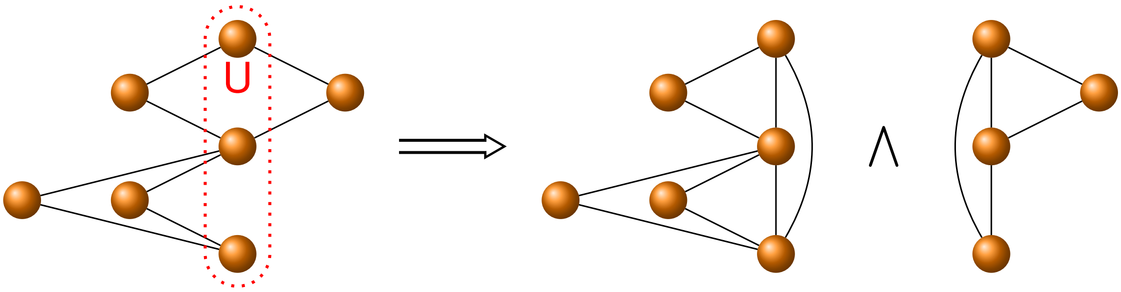

Figure 3.3: Failure state corresponding to a success state after splitting for $|U| = 3 > k = 2$.

Theorem 3.18 (Cutsets of cardinality at most k) *Let $F \subseteq E$ be a cut in G.*
Then for $|F| < k$ it holds:

$$R_{k-ec}(G) = 0.$$

For $|F| = k$, let U_1 and U_2 denote the multisets of vertices of V_1 and V_2 respectively, which are incident to edges of F. Let v denote a new vertex and $F_i := \{\{v,u\}|u \in U_i\}$ with $p_f := 1$ for all $f \in F_i$ new failure-free edge-multisets incident to v and the vertices of U_i, $i \in \{1,2\}$. It holds:

$$R_{k-ec}(G) = \prod_{f \in F} p_f \cdot R_{k-ec}(G_1 + v + F_1) \cdot R_{k-ec}(G_2 + v + F_2).$$

Proof: If $|F| < k$, the graph G is not k-edge connected and therefore the k-edge connected reliability is zero.
If $|F| = k$, we have to show the following:

- G is k-edge connected if and only if all edges of F are intact and for every vertex of V_1 (V_2) there exist edge-disjoint paths to all vertices of U_1 (U_2 respectively) in G_1 (G_2) with their corresponding multiplicity (a vertex $u \in U_1$ (U_2) is assumed to have edge-disjoint paths to itself in arbitrary multiplicity).
- For every vertex of V_1 (V_2) there exist edge-disjoint paths to all vertices of U_1 (U_2 respectively) in G_1 (G_2) with their corresponding multiplicity if and only if $G_1 + v +$

F_1 ($G_2 + v + F_2$) is k-edge connected.

Since the edge sets of E_1, E_2 and F are disjoint, the theorem will follow immediately. Assume $\emptyset \neq H \subseteq F$ is a subset of failing edges. Then $F \setminus H$ is an edge-cut of cardinality $k - |H| < k$ and therefore the graph is not k-edge connected. Hence, for G to be k-edge connected, all edges of F need to be in operating state.
Consider a vertex of V_1, denoted by x. Consider y as the vertex resulting by merging all vertices of V_2. G clearly can only be k-edge-connected, if there exist k edge-disjoint paths between x and y. Since these paths each contain an edge of F, there can only exist k-edge-disjoint paths if there exist edge-disjoint paths between x and all vertices of U_1 in their respective multiplicity. Since the vertex x was chosen arbitrary, this needs to hold for all $x \in V_1$.
We will now show the opposite direction. First consider $x \in V_1$, $y \in V_2$ chosen arbitrary. Then there exist edge-disjoint paths between x to all vertices of U_1 and between y and all vertices of U_2. For each edge $e = \{u_1, u_2\}$ of F, we generate the following path between x and y: Take a path x, u_1 not previously taken, the edge e and a path u_2, y not previously taken. Since there exist edge-disjoint paths between x and U_1 (y and U_2) with respective multiplicities, we can construct our paths in this way. Since E_1 and E_2 are disjoint, we get k-edge disjoint paths between x and y. Hence, x and y are in the same k-edge connected component. This holds for all pairs of x and y. Since being in a k-edge connected component together is an equivalence relation, all vertices of V_1 and V_2 are in the same k-edge connected component.
Now for the second part, assume that for each vertex $x \in V_1$, there exist edge-disjoint paths to all vertices of U_1. By prolonging each of those paths by $\{u, v\} \in F_1$, where u is the corresponding vertex of U_1, we get k-edge disjoint paths between x and v. Hence, $G_1 + v + F_1$ is k-edge connected. Now assume $G_1 + v + F_1$ is k-edge connected. Then there need to be k edge-disjoint paths between x and every vertex $v \in V_1$. If we shorten these paths by $\{u, v\} \in F_1$, we get edge-disjoint paths between x and all vertices of U_1 in their respective multiplicity. □

4 Special graph classes

4.1 Trees, cycles and wheels

Since for a tree G every edge is a bridge and therefore the path between all pairs of vertices is unique, for all $K \subseteq V$, it holds: $R_K^2(G) = 0$.
If $G = C_n$ is a cycle with n vertices, then there exist exactly two paths between all pairs of vertices, which contain all edges of C_n. Hence, independent of $K \subseteq V$, the graph is two-connected, if and only if all edges remain operating. We get: $R_K^2(G) = \prod_{e \in E} p_e$.

Theorem 4.1 *Let $G = W_n$ be the wheel graph with $n+1$ vertices. It holds:*

$$R^2(W_n, p) = \quad p^n(1-(1-p)^n - n \cdot p \cdot (1-p)^{n-1}) + n \cdot (1-p) \cdot p^{n+1}$$

Proof: The graph remains biconnected in the following disjoint cases:

- All outer edges remain intact and at least two edges towards the inner vertex are intact.

$$\underbrace{p^n}_{\text{all outer edges intact}} \cdot \underbrace{(1-(1-p)^n - n \cdot p \cdot (1-p)^{n-1})}_{\text{at least two inner edges intact}}$$

- Exactly one outer edge $e = \{u, w\}$ fails and the edges $\{u, v\}$ and $\{w, v\}$ are intact.

$$\underbrace{n \cdot (1-p) \cdot p^{n-1}}_{\text{one outer edge fails}} \cdot \underbrace{p^2}_{\text{the two inner edges intact}}$$

If more than one outer edge fails, v is an articulation and hence the graph is not biconnected. □

4.2 Series-parallel graphs and two-trees

Theorem 4.2 *Let $T_{n,k}^2$ be a two-tree as defined in Chapter 1. Then all edges selected in the procedure are irrelevant, all other edges are essential and hence*

$$R^2(T_{n,k}^2, p) = p^{n+k}.$$

Proof: First, we will proof the theorem for simple two-trees.
For $K_3 = T_{3,0}^2$, the theorem holds since $R^2(K_3, p) = p^3$. Note, that all edges are essential. Now consider a simple two-tree $T_{n,0}^2$ and the graph $T_{n+1,0}^2$ generated by selecting a

previously not selected edge e and attaching a new vertex v to it. By induction the theorem holds for $T^2_{n,0}$ and e is essential, hence

$$p^n = R^2(T^2_{n,0}, p) = p \cdot R^2(T^2_{n,0}|_{p_e=1}).$$

The vertex v has degree two. By applying Theorem 3.5 to v and afterwards Theorem 3.2 to e and the new parallel edge, we get

$$R^2(T^2_{n+1,0}, p) = p^2 \cdot R^2(T^2_{n,0}|_{p_e=1}) = p^{n+1}.$$

Now consider a two-tree $T^2_{n,k}$ with $k > 0$ and assume the theorem to be true for all $k' < k$. Denote by $e = \{u, w\}$ the last edge which was selected multiple times. Then the end vertices of e are a separator of cardinality two (hence e is irrelevant), where we choose V_1 and V_2 such that $T^2_{n,k}{<}V_2{>}$ is the simple two-tree added to e in the procedure. Using Theorem 3.11 we get

$$R^2(T^2_{n,k}, p) = R^2(G_1 + f) \cdot R^2(G_2 + f)$$

where $G_2 = T^2_{n,k}{<}V_2{>} = T^2_{n'',0}$, $G_1 = T^2_{n,k}{<}V_1{>} = T^2_{n',k-1}$, $f = \{u, w\}$ with $p_f = 1$ and $n' + n'' = n + 2$. By induction, e is essential in G_2 and irrelevant in G_1. By Theorem 3.2 and induction, we get

$$\begin{aligned} R^2(G_1 + f) = R^2(G_1|_{p_e=1}) &= \quad p^{n'+k-1} \quad \text{and} \\ R^2(G_2 + f) = R^2(G_2|_{p_e=1}) &= \quad p^{n''-1}. \end{aligned}$$

With $n' + n'' = n + 2$ the theorem follows. □

Theorem 4.3 *Let $G = (V, E)$ be a biconnected series-parallel graph with $n = |V|$ vertices. Then, there exists an integer k with $0 \le k \le n - 3$ such that:*

$$R^2(G, p) = p^{n+k}.$$

Proof: Since maximal series-parallel graphs are two-trees [dF01] [WC83], consider the two-tree $T^2_{n,k}$ resulting from G by addition of edges. Let F denote the set of essential edges of $T^2_{n,k}$ ($|F| = n+k$). Because G is biconnected, $F \subseteq E$. Hence, F is a success set of G and $R^2(G, p) \ge p^{n+k}$. Since G is a spanning subgraph of $T^2_{n,k}$ it holds $R^2(G, p) \le R^2(T^2_{n,k}, p) = p^{n+k}$. We obtain $R^2(G, p) = p^{n+k}$. Since $T^2_{n,k}$ has $2n - 3$ edges (given by the procedure used to create two-trees) and G has at least n edges to be biconnected, $0 \le k \le n - 3$. Similar results applied to edge sets can be found in [BR11]. □

4.3 Complete graph K_n

For the all-terminal reliability polynomial of the complete graph, Gilbert [Gil59] presented the following recurrence relation:

Theorem 4.4 (Gilbert, 1959) *For the all-terminal reliability polynomial of the complete graph, for short denoted by $R(K_n)$, holds:*

$$1 = \sum_{k=1}^{n} \binom{n-1}{k-1} \cdot R(K_k) \cdot q^{k\cdot(n-k)}.$$

Proof: We consider a fixed vertex $v \in V$ and its connected component after edge failure. The probability that v is in a connected component K of size k is

$$\underbrace{\binom{n-1}{k-1}}_{\text{choice of vertices for } K} \cdot \underbrace{R(K_k)}_{\text{K is connected}} \cdot \underbrace{q^{k\cdot(n-k)}}_{\substack{\text{edges between } K \\ \text{and } V\setminus K \text{ fail}}}.$$

The vertex v is in exactly one connected component. Hence, if we sum those probabilities for $k = 1, \ldots, n$, we get the certain event, which has probability one. □

Since $R(K_k)$ for all $k \leq n$ can be calculated in time $\mathcal{O}(n^2)$ by this recurrence relation, we will assume that $R(K_k)$ is known and use a similar approach to calculate the biconnected reliability polynomial.

4.3.1 A recurrence relation for the biconnected reliability polynomial

We investigate the event that after an edge failure the graph remains connected. We consider a fixed vertex v and partition the remaining vertex set depending on their connectivity to v (see Figure 4.1):

- Let E denote the bridges of G incident to v. Let I denote the connected component containing v in $G - E$. Then $S := V(G) \setminus V(I)$.
- The vertex set K contains all vertices of the block of v (if v is contained in several blocks, one arbitrarily chosen one is fixed for K)
- Let H denote the connected component containing v in $G - (K \setminus \{v\})$. Then $T := V(H) \setminus S$ and $L := V(G) \setminus (V(H) \cup K)$

Given the partition of V into these vertex sets, we can derive independent formulas for the probabilities of those sets. For the biconnected reliability it then remains to sum over all possible partitions of V and take into account that all edges between S, L and T and between S, T and $K \setminus \{v\}$ must fail.

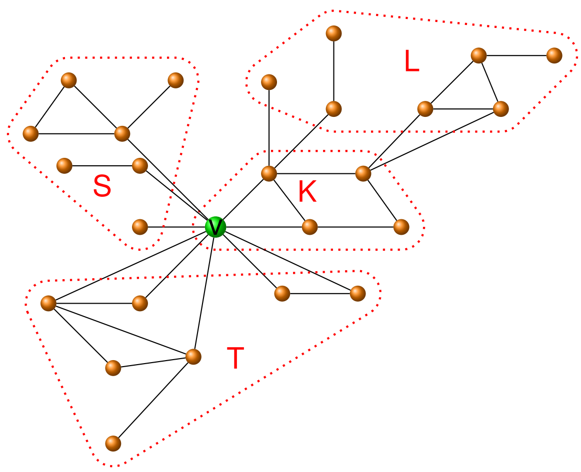

Figure 4.1: Partitioning of $V(G)$ depending on connectivity to fixed vertex v

Remark 4.5 Due to the symmetry it suffices to consider the size of the partition rather then the specific vertex sets.

Lemma 4.6 *Let s be a non-negative integer and $r = s + 1$. We consider a complete graph K_r with vertex set $\{1, \dots, s, v\}$ and independently with probability q failing edges. The probability that the remaining graph is connected after edge failure and all edges incident to v are bridges is denoted by $p_1(s)$. It holds:*

$$p_1(s) = \sum_{\lambda \vdash s} \left\{ \begin{matrix} s \\ \lambda \end{matrix} \right\} \cdot (1-q)^{|\lambda|} \cdot q^{s-|\lambda|} \cdot \prod_{i=1}^{|\lambda|} R(K_{\lambda_i}) \cdot \lambda_i \cdot q^{\lambda_i (s-\lambda_i)/2}$$

Remark 4.7 We will use Lemma 4.6 to describe vertex set S.

Proof: Assume that H is a random connected spanning subgraph of K_r such that $H - v$ has exactly j components and $\deg v = j$. Then the vertex sets of these components form a partition of the set $\{1, \dots, s\}$ with j blocks. Figure 4.2 illustrates the structure of H, where the bubbles represent connected subgraphs. There are $\left\{ \begin{matrix} s \\ \lambda \end{matrix} \right\}$ ways to partition the set $\{1, \dots, s\}$ such that the resulting set partition has $j = |\lambda|$ blocks of sizes $\lambda_1, \dots, \lambda_j$. The probability that a block of size λ_i forms a connected subgraph is $R(K_{\lambda_i})$. All edges between different blocks must fail. The probability of failure of all edges that connect a vertex of block i with a vertex of any other block of the partition is $q^{\lambda_i (s-\lambda_i)}$. Considering all blocks, we count each edge between blocks twice - once for each of the incident blocks. Every block has exactly one intact edge incident to v, where we have λ_i possible selections for the vertex of the block. Hence we get the probability

$$\prod_{i=1}^{|\lambda|} (1-q) \cdot q^{\lambda_i - 1} \cdot \lambda_i.$$

Composing these subresults yields

$$p_1(s) = \sum_{\lambda \vdash s} \left\{ {s \atop \lambda} \right\} \prod_{i=1}^{|\lambda|} R(K_{\lambda_i}) \cdot (1-q) \cdot q^{\lambda_i - 1} \cdot \lambda_i \cdot q^{\lambda_i (s-\lambda_i)/2}$$

which transforms to the statement of the lemma by using $\sum_i \lambda_i = s$ □

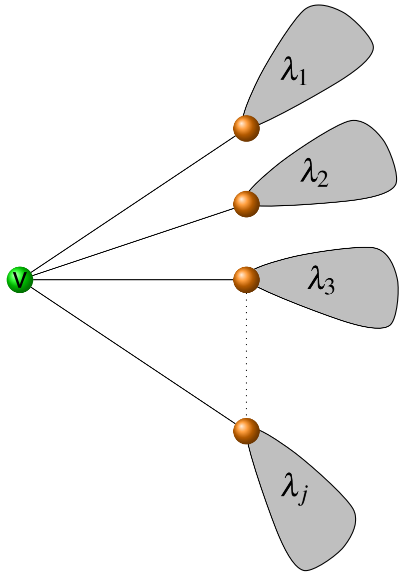

Figure 4.2: All edges incident to v are bridges.

Lemma 4.8 *Let $E_k = (X = \{x_1, \ldots, x_{k-1}, v\}, \emptyset)$ be an empty graph with at least three vertices ($k \geq 3$) and $K_l = (Y, F)$ a complete graph of order l, $l \geq 0$, whose vertex set is disjoint from X, i.e. $X \cap Y = \emptyset$. We consider the join $G_{k,l} = E_k + K_l$. The edges of $G_{k,l}$ are assumed to fail statistically independent with identical probability q. Let $p_2(k,l)$ be the probability that $G_{k,l}$ decomposes into exactly k connected components, such that no vertices of X are in the same component and v is an isolated vertex. Then it holds:*

$$p_2(k,l) = \sum_{\lambda \vdash l} (k-1)^{\underline{|\lambda|}} \left\{ {l \atop \lambda} \right\} \cdot q^{l \cdot (k-1)} \cdot \prod_{i=1}^{|\lambda|} R(K_{\lambda_i + 1}) \cdot q^{\lambda_i (l - \lambda_i)/2}$$

Remark 4.9 We will use Lemma 4.8 to account for all vertices connected to v via articulations in $K \setminus \{v\}$. These vertices are denoted by the vertex set L.

Proof: Denote with $X' = \{x'_1, \ldots, x'_r\}$ a subset of X containing all vertices which are not isolated. Then there must be a partition $\pi = \{Y_1, \ldots, Y_r\}$ of Y such that the graphs induced by $Y_i \cup \{x'_i\}$ are connected for $i = 1, \ldots, r$. Additionally we require that these r induced graphs form r separate components. This decomposition is depicted schematic in Figure 4.3.

First, we select a set partition π of type λ of Y for which we have $\left\{ {l \atop \lambda} \right\}$ possibilities.

Then we choose $|\lambda|$ out of $k-1$ vertices from X. There are $\binom{k-1}{|\lambda|}$ ways to select these vertices. Let X' be the chosen subset of X. There are $|\lambda|!$ bijections $\phi : \pi \to X'$ that assign one vertex to each block of π, which gives

$$\binom{k-1}{|\lambda|} |\lambda|! = (k-1)^{\underline{|\lambda|}}$$

possibilites to form the vertex sets of the $r = |\lambda|$ components. There can be no edge between different components within K_l, which occurs with probability

$$\prod_{i=1}^{|\lambda|} q^{\lambda_i(l-\lambda_i)/2}$$

We also have to exclude edges between a vertex $y \in Y_i$ and x_j whenever $\phi(Y_i) \neq x_j$, which provides the factor $q^{l(k-1)}$. Finally

$$\prod_{i=1}^{|\lambda|} R(K_{\lambda_i+1})$$

yields the probability that all induced subgraphs are connected. □

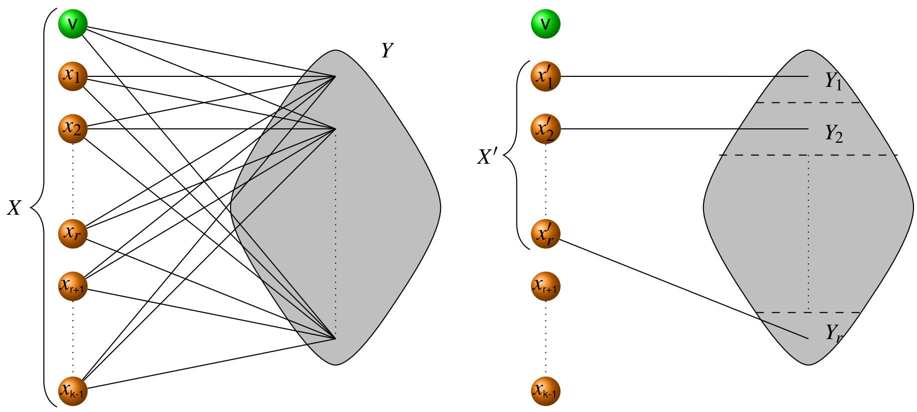

Figure 4.3: Decomposition of $G_{k,l}$ into connected components after edge failure

Lemma 4.10 *Let t be a non-negative integer. Consider the complete graph with vertex set $\{1,\ldots,t,v\}$ and independently with probability q failing edges and let G denote the remaining graph after edge failure. Let $p_3(t,b)\cdot(b+1)$ denote the probability that $G-v$ has exactly b connected components where each has at least two incident edges to v in G. Further, let $p_3(t) = \sum_b p_3(t,b)$. Then*

$$p_3(t) = \sum_{\sigma \vdash (t,2)} \frac{1}{|\sigma|+1} \left\{ {t \atop \sigma} \right\} \prod_{i=1}^{|\sigma|} R(K_{\sigma_i}) \left(1 - q^{\sigma_i} - \sigma_i \cdot pq^{\sigma_i - 1}\right) \cdot q^{\sigma_i(t-\sigma_i)/2}.$$

Remark 4.11 We will use Lemma 4.10 to describe the probability that t vertices are connected to v via b additional blocks containing v besides K. The factor $(b+1)$ accounts for the choice of K out of those $(b+1)$ blocks.

Proof: Consider a fixed partition of $\{1,\dots,t\}$, such that each block contains at least two vertices and let t_i denote the size of the ith block, $i=1,\dots,b$. The t_i vertices form a connected component in $G-v$ and at least two edges towards v must remain intact. Additionally, the edges between different blocks must fail. Figure 4.4 illustrates this event. The probability for this event is:

$$p_3(b,t,\{t_i\}) := \prod_{i=1}^{b} \underbrace{R(K_{t_i})}_{\substack{t_i \text{ connected} \\ \text{component}}} \cdot \underbrace{\left(1-q^{t_i}-t_i\cdot(1-q)q^{t_i-1}\right)}_{\substack{\text{at least 2 edges} \\ \text{to } v \text{ intact}}} \cdot \underbrace{q^{t_i\cdot(t-t_i)/2}}_{\substack{t_i \text{ disconnected from} \\ \text{all other blocks}}}$$

Since until now we considered a fixed assignment of t. We now have to sum over all possible assignments and consider permutations which result in the same graph.

$$p_3(b,t)\cdot(b+1) = \underbrace{\frac{1}{b!}}_{\substack{\text{permutations} \\ \text{of the } t_i}} \cdot \sum_{\substack{\sum_{i=1}^{b} t_i=|t| \\ t_i\geq 2}} \underbrace{\binom{t}{t_1,\dots t_t}}_{\substack{\text{number of assignments} \\ \text{of } t \text{ towards } \{t_i\}}} \cdot p_3(b,t,\{t_i\})$$

By changing from the sum over all sums to number partitions where each block has at least size two we get

$$\begin{aligned} p_3(b,t) &= \frac{1}{b+1} \sum_{\substack{\sigma\vdash(t,2) \\ |\sigma|=b}} \left\{ \begin{matrix} t \\ \sigma \end{matrix} \right\} p_3(b,t,\{\sigma_i\}) \\ &= \frac{1}{b+1} \sum_{\substack{\sigma\vdash(t,2) \\ |\sigma|=b}} \left\{ \begin{matrix} t \\ \sigma \end{matrix} \right\} \prod_{i=1}^{|\sigma|} R(K_{\sigma_i})\cdot\left(1-q^{\sigma_i}-\sigma_i\cdot(1-q)q^{\sigma_i-1}\right)\cdot q^{\sigma_i\cdot(t-\sigma_i)/2}. \end{aligned}$$

Hence, we get

$$\begin{aligned} p_3(t) &= \sum_b p_3(t,b) \\ &= \sum_b \frac{1}{b+1} \sum_{\substack{\sigma\vdash(t,2) \\ |\sigma|=b}} \left\{ \begin{matrix} t \\ \sigma \end{matrix} \right\} \prod_{i=1}^{|\sigma|} R(K_{\sigma_i})\cdot\left(1-q^{\sigma_i}-\sigma_i(1-q)q^{\sigma_i-1}\right) q^{\sigma_i\cdot(t-\sigma_i)/2} \\ &= \sum_{\sigma\vdash(t,2)} \frac{1}{|\sigma|+1} \left\{ \begin{matrix} t \\ \sigma \end{matrix} \right\} \prod_{i=1}^{|\sigma|} R(K_{\sigma_i})\cdot\left(1-q^{\sigma_i}-\sigma_i(1-q)q^{\sigma_i-1}\right) q^{\sigma_i\cdot(t-\sigma_i)/2}. \qquad \square \end{aligned}$$

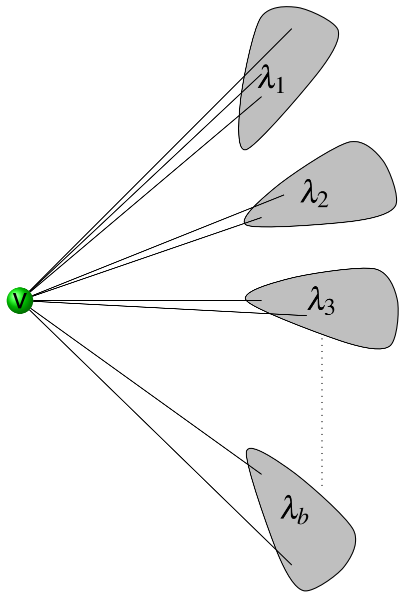

Figure 4.4: Each component of $G-v$ contains exactly one block of G containing v

Theorem 4.12 *For the biconnected reliability polynomial of the complete graph the following recurrence relation holds:*

$$\begin{aligned} R(K_n) = & p_1(n-1) \\ & + \sum_{k=3}^{n} \binom{n-1}{k-1} R^2(K_k) \sum_{t=0}^{n-k} \binom{n-k}{t} \cdot p_3(t) \\ & \cdot \sum_{s=0}^{n-k-t} \binom{n-k-t}{s} p_1(s) \cdot p_2(k, n-k-t-s) \cdot q^c \end{aligned}$$

with $c = (n-k-t-s)\cdot(s+t) + s\cdot(t+k-1) + t\cdot(k-1)$

Proof: We consider all possibilites for resulting graphs after edge failure, where the remaining graph is still connected. If our vertex v is not part of any block, all edges incident to v must be bridges. This is described by $p_1(n-1)$. Else, v is part of at least one block consisting of $k \geq 3$ vertices. Then we divide our vertex set into the disjoint vertex sets described in the beginning of this section, with $k = |K|, t = |T|, s = |S|, |L| = (n-k-t-s)$ (the remaining vertices).

For the block of size k we have $\binom{n-1}{k-1}$ ways to pick the additional vertices and $R^2(K_k)$ describes the chance that those build a block. Afterwards we have $\binom{n-k}{t}$ ways to pick the vertices for T. Then we have $\binom{n-k-t}{s}$ ways to choose the vertices for S. All other vertices must belong to L, so there is no choice. The probability for the connection of the vertex sets to v is then described by p_1, p_3 and p_2 respectively. The terms p_1, p_2 and p_3 only consider edge failure inside the given sets. Hence, we get an additional

n	$bc(n)$
3	1
4	10
5	238
6	11368
7	1014888
8	166537616
9	50680432112
10	29107809374336
25	2036960093236377377478641609563926626010214566222803003806046
	18049840134502972580111 0044672
55	1070393741162968413303752936465051181837612231776049039650437
	2019602883705926408936111634369909406169405132959503910388758
	7841259299554870101283192072709038105936318651700868299691279
	9362149555009021273839081685946443729617017492974793909864594
	4707240482191228191907372925474115351225309958153732703179699
	0560984395645956594775148060369979283153664597086859221299829
	8394862641141580779835059315690113549194527952757822912840313
	2616714228007604060 16

Table 4.1: Number of biconnected graphs on n vertices

factor describing the edge failures between S, T and L as well as S, T and $K \setminus \{v\}$: q^c with $c = |L| \cdot s + |L| \cdot t + s \cdot t + s \cdot (k-1) + t \cdot (k-1)$. □

4.3.2 Counting biconnected graphs

The recurrence relation for the complete graph can be used to calculate the number of biconnected graphs on n vertices:

Theorem 4.13 *The number of labelled biconnected graphs $bc(n)$, listed in OEIS A013922, can be calculated by:*

$$bc(n) = 2^{\frac{n \cdot (n-1)}{2}} \cdot R^2(K_n, 0.5)$$

Proof: If we set $p = q = 0.5$, every graph with n vertices arises with the same probability $\frac{1}{2^{\frac{n \cdot (n-1)}{2}}}$. Hence, if we multiply $R^2(K_n, 0.5)$ with $2^{\frac{n \cdot (n-1)}{2}}$, each biconnected graph is counted exactly once. □

While a representation via an exponential generating function was presented by Harary and Palmer [HP73] we used Theorem 4.13 to calculate $bc(n)$ for up to 55 vertices, with the results shown in Table 4.1.

4.3.3 Running time analysis

For the running time analysis we assume that a^n can be calculated in a single time step. Hardy and Ramanujan [HR18] and Uspensky [Usp20] showed, that the number of partitions of n, $p(n)$, is asymptotically

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

Hence, it follows $p(n) \in \mathcal{O}\left(\frac{13,002^{\sqrt{n}}}{n}\right)$. By applying this to our recurrence relation for $R^2(K_n)$ we get the following theorem:

Theorem 4.14 *$R^2(K_n)$ can be calculated via Theorem 4.12 in time $\mathcal{O}(n^2 \cdot 13,002^{\sqrt{n}})$ and space $\mathcal{O}(n^6)$.*

Proof: To calculate $R^2(K_n)$ via Theorem 4.12 we need to calculate the following subresults and store them:

- $R(K_i)$ for $i = 1,\ldots,n$
- $p_1(s)$ for $s = 1,\ldots,n$
- $p_2(k,l)$ for $k = 1,\ldots,n$, $l = 1,\ldots,n$
- $p_3(t)$ for $t = 1,\ldots,n$

Afterwards, to calculate $R^2(K_i)$ for $i \in \{1,\ldots,n\}$ we need to sum over all choices for k, t, s and l which can be done in time $\mathcal{O}(n^4)$. So to calculate all $R^2(K_i)$ after calculating and storing our subresults (runtimes listed in table 4.2) we need $\mathcal{O}(n^5)$ additional operations. $R(K_i)$ for $i = 1,\ldots,n$ can be calculated in time $\mathcal{O}(n^2)$ via the recurrence relation presented by Gilbert [Gil59]. To calculate $p_1(s)$ for a given $s \leq n$ we need to sum over all partitions and consider every block of the partition, which can be done in $\mathcal{O}\left(\frac{13,002^{\sqrt{s}}}{s} \cdot s\right)$ calculations. To store the result for a certain value of s as polynomial, we have a polynomial of degree at most $\frac{s \cdot (s+1)}{2}$ and since every edge can be intact or can fail, which each may change coefficients by ± 1, the highest coefficient has at most length $\mathcal{O}\left(s^2\right)$. Therefore $p_1(s)$ can be stored in space $\mathcal{O}\left(s^4\right)$.
By the same reasoning, we get that $p_2(k,l)$ for given k and l can be calculated in time $\mathcal{O}\left(13,002^{\sqrt{l}}\right)$ and space $\mathcal{O}\left(l^2 \cdot \max\{k,l\}^2\right)$.
For $p_3(t)$, the number of summands of $\sum_{\sigma \vdash (t,2)}$ is bounded above by $p(t)$. Hence the runtime for given t is in $\mathcal{O}\left(13,002^{\sqrt{t}}\right)$. Since, due to the term $\frac{1}{b+1}$ we can not guarantee that the coefficients are integers, we could store the polynomial times $(b+1)$ for each value of b individually which therefore can be done in space $\mathcal{O}(t^5)$.

Since s, k, l and t are all bounded above by n, all necessary subresults can be calculated in time $\mathcal{O}\left(n^2 \cdot 13{,}002^{\sqrt{n}}\right)$ and space $\mathcal{O}(n^6)$.
Since the summation over all subresults can be done in polynomial time and storing the biconnected reliability polynomial for $R^2(K_k)$ for given k can be done in space $\mathcal{O}(k^4)$, the run-time of the complete algorithm is in $\mathcal{O}\left(n^2 \cdot 13{,}002^{\sqrt{n}}\right)$ and the space needed is in $\mathcal{O}(n^6)$. □

Result	runtime	space	# values
$R(K_i)$	$\mathcal{O}(i)$	$\mathcal{O}(i^4)$	n
$p_1(s)$	$\mathcal{O}(13{,}002^{\sqrt{s}})$	$\mathcal{O}(s^4)$	n
$p_2(k,l)$	$\mathcal{O}(13{,}002^{\sqrt{l}})$	$\mathcal{O}(l^2 \cdot \max\{k,l\}^2)$	n^2
$p_3(t)$	$\mathcal{O}(13{,}002^{\sqrt{t}})$	$\mathcal{O}(t^5)$	n

Table 4.2: Runtime and space requirement of the different subfunctions

4.3.4 Two-edge reliability polynomial

For the two-edge reliability polynomial the following recurrence relation was presented by Reinwardt [Rei15]:

Theorem 4.15 *For the two-edge reliability of the complete graph K_n, for short denoted by r_n, holds:*

$$r_n = 1 - \sum_{\lambda \vdash n} \left\{ {n \atop \lambda} \right\} \prod_{i=1}^{|\lambda|} q^{\frac{\lambda_i(n-\lambda_i)}{2}} \sum_{\substack{\sigma \vdash \lambda_i \\ \sigma \neq (n)}} \left\{ {\lambda_i \atop \sigma} \right\} (1-q)^{|\sigma|-1} q^{1-|\sigma|} t_\sigma \prod_{j=1}^{|\sigma|} q^{\frac{\sigma_j(\lambda_i-\sigma_j)}{2}} r_{\sigma_j}$$

where t_σ denotes the number of spanning trees of λ_i connecting the components of σ.

Proof: The proof is by considering all events (therefore "$1-$" after solving for r_n) of edge failure and considering all connected components (λ_i) and their respective two-edge connected components (σ_j) and calculating the probability for those. For the full proof, see Reinwardt [Rei15]. □

By limiting to the event that the resulting graph is connected, this formula simplies while the given proof by Reinwardt still holds:

Theorem 4.16 *For the two-edge reliability polynomial of K_n, $R_{2-ec}(K_n)$, holds:*

$$R_{2-ec}(K_n) = R(K_n) - \sum_{\substack{\sigma \vdash n \\ \sigma \neq (n)}} \left\{ {n \atop \sigma} \right\} (1-q)^{|\sigma|-1} q^{1-|\sigma|} t_\sigma \prod_{j=1}^{|\sigma|} q^{\frac{\sigma_j(n-\sigma_j)}{2}} R_{2-ec}(K_{\sigma_j})$$

Alternatively we can as well again consider a fixed vertex $v \in V$ and get the following recurrence relation:

Theorem 4.17 *For the two-edge reliability polynomial of K_n the following recurrence relation holds:*

$$R(K_n) = \sum_{k=1}^{n} \binom{n-1}{k-1} R_{2-ec}(K_k) \cdot q^{k(n-k)} \cdot \sum_{\lambda \vdash n-k} \left\{ {n-k \atop \lambda} \right\} \left(\frac{1-q}{q} \right)^{|\lambda|} \cdot \prod_{i=1}^{|\lambda|} R(K_{\lambda_i}) \cdot k \cdot \lambda_i \cdot q^{\lambda_i(n-k-\lambda_i)/2}.$$

Proof: We consider a fixed vertex $v \in V$ and the event that the graph G, resulting after edge failure, remains connected. Then v is in a two-edge connected component K of size $k \geq 1$. Consider the connected components of $G - K$. They form a partitioning of the vertex set $V \setminus K$ with block size distribution λ, which is illustrated in Figure 4.5. For each of the blocks, the following properties hold:

- the block is connected
- exactly one edge to K is intact in G
- edges between different blocks fail

Hence, for the ith block we get the following probability independent of the other blocks:

$$\underbrace{R(K_{\lambda_i})}_{\text{block connected}} \cdot \underbrace{k \cdot \lambda_i \cdot (1-q) \cdot q^{k \cdot \lambda_i - 1}}_{\text{exactly 1 edge to K}} \cdot \underbrace{q^{\lambda_i(n-k-\lambda_i)/2}}_{\text{edges to other blocks fail}}$$

We have to sum over all block size distributions of set partitions of the vertex set $V \setminus K$, hence

$$\sum_{\lambda \vdash n-k} \underbrace{\left\{ {n-k \atop \lambda} \right\}}_{\substack{\text{number of set partitions} \\ \text{with block size distribution } \lambda}} \prod_{i=1}^{|\lambda|} R(K_{\lambda_i}) \cdot k\, \lambda_i \cdot (1-q) \cdot q^{k \cdot \lambda_i - 1} \cdot q^{\lambda_i(n-k-\lambda_i)/2}$$

$$= \sum_{\lambda \vdash n-k} \left\{ {n-k \atop \lambda} \right\} \cdot (1-q)^{|\lambda|} \cdot q^{k \cdot (n-k) - |\lambda|} \cdot \prod_{i=1}^{|\lambda|} R(K_{\lambda_i}) \cdot k\, \lambda_i \cdot q^{\lambda_i(n-k-\lambda_i)/2}.$$

The probability for K to be the two-edge-connected component of v is given by

$$\underbrace{\binom{n-1}{k-1}}_{\text{choices for K}} \cdot \underbrace{R_{2-ec}(K_k)}_{K \text{ two-edge-connected}} .$$

Summing over all possible sizes k of K yields the considered event that the graph is connected and therefore $R(K_n)$ and the theorem holds.
It is noteworthy that the recurrence relation gives $R_{2-ec}(K_1) = 1$ and $R_{2-ec}(K_2) = 0$,

hence, since the λ_i are connected to K via bridges, unlike for the biconnected reliability, we do not need a special case in the event that all edges incident to v form bridges. □

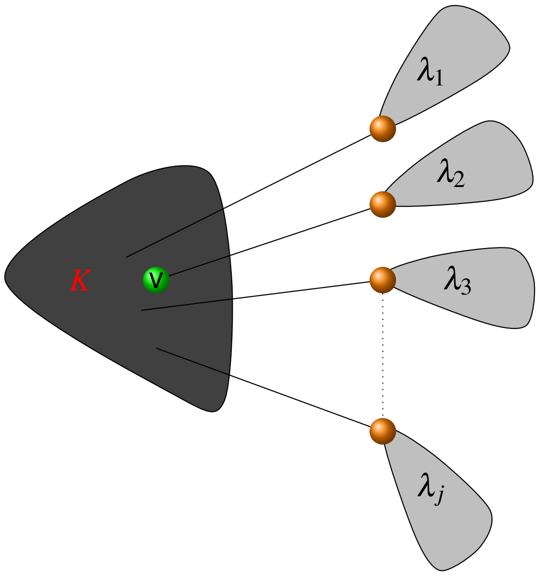

Figure 4.5: Partitioning of $V(K_n)$ into the two-edge connected component K containing v and connected components λ_i connected to v via bridges

4.4 Complete bipartite graphs $K_{a,b}$

For certain (small) choices of a, we can derive explicit formulas for $K_{a,b}$ by analyzing what different vertex degree distributions of B after edge failure result in biconnected graphs.
It follows immidiatly, that

$$R^2(K_{1,b}) = 0 \quad \text{and}$$
$$R^2(K_{2,b}) = p^{2b}.$$

4.4.1 Complete bipartite graphs $K_{3,b}$

Theorem 4.18 *For the complete bipartite graph $K_{3,b}$, $b \geq 3$ holds*

$$R^2(K_{3,b}) = (p^3 + 3qp^2)^b - 3 \cdot b \cdot p^3 (qp^2)^{b-1} - (qp^2)^b \cdot (3 \cdot 2^b - 3).$$

Proof: Denote the vertices of A by u, v, w. For the remaining graph G to be biconnected, every vertex of B need to have at least degree 2. The degree-2-vertices of B can be partitioned into three classes depending on their neighbourhood. Then in the following disjoint cases the graph is biconnected:

1. At least two vertices of B have degree 3.
2. Exactly one vertex of B has degree 3 and every vertex of A is adjacent to at least one degree-2-vertex of B.
3. All vertices of B have degree 2 and all three classes of degree-2-vertices are prevalent.

Case 1
The probability for this case is: $P_1 := \sum_{k=2}^{b} \binom{b}{k} \left(p^3\right)^k \cdot \left(3qp^2\right)^{b-k}$, where k denotes the number of vertices of degree 3 of B.
By applying the Binomial theorem, we get

$$P_1 = \left(p^3 + 3qp^2\right)^b - b \cdot p^3 \left(3qp^2\right)^{b-1} - \left(3qp^2\right)^b.$$

Case 2
The probability for this case is

$$P_2 := b \cdot p^3 \cdot \left(qp^2\right)^{b-1} \cdot (3^{b-1} - 3)$$

where the $(3^{b-1} - 3)$ regards that all assignments of the degree-2-vertices in the three subclasses are valid, unless all vertices get assigned to the same class.
Case 3
The probability for this case is

$$P_3 := \left(qp^2\right)^b \cdot (3^b - 3 \cdot 2^b + 3),$$

where the factor $(3^b - 3 \cdot 2^b + 3)$ accounts for an arbitrary assignment towards the three classes such that all are non-empty.

The theorem then follows by $R^2(K_{3,b}) = P_1 + P_2 + P_3$. □

4.4.2 Complete bipartite graphs $K_{4,b}$

Theorem 4.19 *For the complete bipartite graph $K_{4,b}$, $b \geq 4$ holds*

$$\begin{aligned} R^2(K_{4,b}) = &\left(p^4 + 4p^3q + 6p^2q^2\right)^b - 4b \cdot p^4 \cdot \left(p^3q + 3p^2q^2\right)^{b-1} \\ &- 3b \cdot p^4 \cdot \left(2p^2q^2\right)^{b-1} + 12b \cdot p^4 \cdot \left(p^2q^2\right)^{b-1} \\ &+ 6b(b-1) \cdot (p^3q)^2 \cdot (p^2q^2)^{b-2} - 12b \cdot p^3q(p^3q + 3p^2q^2)^{b-1} \\ &+ 24b \cdot p^3q \left(2p^2q^2\right)^{b-1} - 12b \cdot p^3q(p^2q^2)^{b-1} - 12(p^3q + 4p^2q^2)^b \\ &+ 8(p^3q + 3p^2q^2)^b + (p^2q^2)^b \cdot (20 \cdot 3^b - 27 \cdot 2^b + 12). \end{aligned}$$

Proof: We again investige all possible vertex degree distributions of B, which result in a biconnected graph.

Case 1: At least two vertices have degree 4.

$$P_1 := \sum_{k=2}^{b} \binom{b}{k} (p^4)^k \cdot (4qp^3 + 6p^2q^2)^{b-k}$$

Case 2: Exactly one vertex has degree 4.

Case 2.1: there are at least two classes of degree-3-vertices which are non-empty.

$$P_{2.1} := b \cdot p^4 \cdot \sum_{k=2}^{b-1} \binom{b-1}{k} \cdot (p^3q)^k \cdot (4^k - 4) \cdot (6p^2q^2)^{b-1-k}$$

Case 2.2: There is exactly one class of degree.3-vertices.

$$P_{2.2} := b \cdot p^4 \cdot \sum_{k=1}^{b-1} \binom{b-1}{k} \cdot 4 \cdot (p^3q)^k \cdot (p^2q^2)^{b-1-k} \underbrace{(6^{b-1-k} - 3^{b-1-k})}_{\substack{\text{reaching missing} \\ \text{vertex by degree-2-vertices}}}$$

Case 2.3: There is no degree-3-vertex.

Then there need to be at least three classes of degree-2-vertices (otherwise the degree-4-vertex is an articulation) and every vertex of A needs to be adjacent to a degree-2-vertex.

$$P_{2.3} := b \cdot p^4 \cdot (p^2q^2)^{b-1} \cdot \left(\underbrace{6^{b-1} - 4 \cdot 3^{b-1} + 6}_{\text{every a-vertex reached}} - \underbrace{(3 \cdot 2^{b-1} - 6)}_{\substack{\text{short-cycles} \\ \text{reaching all a-vertices}}} \right)$$

Case 3: There are at least three classes of degree-3-vertices.

$$P_3 := \sum_{k=3}^{b} \binom{b}{k} (p^3q)^k \cdot (4^k - 6 \cdot 2^k + 8) \cdot (6p^2q^2)^{b-k}$$

Case 4: There are exactly two classes of degree-3-vertices.

Case 4.1: Both classes have at least two vertices.

$$P_{4.1} := \sum_{k=4}^{b} \binom{b}{k} \cdot 6 \cdot (p^3q)^k \cdot (2^k - 2 - 2k) \cdot (6p^2q^2)^{b-k}$$

Case 4.2: One class has exactly one vertex.

The a-vertex adjacent to only one degree-3-vertex then needs to be adjacent to a degree-2-vertex.

$$P_{4.2} := \sum_{k=3}^{b} \binom{b}{k} \cdot 4 \cdot 3 \cdot k \cdot (p^3q)^k \cdot (p^2q^2)^{b-k} \cdot (6^{b-k} - 3^{b-k})$$

Case 4.3: Both classes have exactly one vertex.

Case 4.3.1: The two a-vertices with only one degree-3-neighbour have a common degree-2-neighbour.

$$P_{4.3.1} := \binom{b}{2} \cdot 4 \cdot 3 \cdot (p^3q)^2 \cdot (p^2q^2)^{b-2} \cdot (6^{b-2} - 5^{b-2})$$

Case 4.3.2: The two a-vertices do not have a common degree-2-neighbour but are both adjacent to at least one degree-2-vertex.

$$P_{4.3.2} := \binom{b}{2} \cdot 4 \cdot 3 \cdot (p^3q)^2 \cdot (p^2q^2)^{b-2} \cdot (5^{b-2} - 2 \cdot 3^{b-2} + 1)$$

We get

$$P_{4.3} := P_{4.3.1} + P_{4.3.2} = \binom{b}{2} \cdot 12 \cdot (p^3q)^2 \cdot (p^2q^2)^{b-2} \cdot (6^{b-2} - 2 \cdot 3^{b-2} + 1).$$

Case 5: There is exactly one class of degree-3-vertices which has at least two vertices. Then the remaining a-vertex has to be adjacent to two different classes of degree-2-vertices.
$P_5 := \sum_{k=2}^{b} \binom{b}{k} \cdot 4 \cdot (p^3q)^k \cdot (p^2q^2)^{b-k}(6^{b-k} - 3 \cdot 4^{b-k} + 2 \cdot 3^{b-k})$

Case 6: There is exactly one degree-3-vertex.

Case 6.1: For the remaining a-vertex all three classes of adjacent degree-2-vertices are non-empty.
$P_{6.1} := 4b \cdot p^3q \cdot (p^2q^2)^{b-1} \cdot (6^{b-1} - 3 \cdot 5^{b-1} + 3 \cdot 4^{b-1} - 3^{b-1})$

Case 6.2: For the remaining a-vertex exactly two classes of adjacent degree-2-vertices are non-empty.
Then the second a-vertex of the missing class must be reached by the remaining degree-2-vertices.
$P_{6.2} := 4b \cdot p^3q \cdot (p^2q^2)^{b-1} \cdot 3 \cdot (5^{b-1} - 2 \cdot 4^{b-1} + 2 \cdot 2^{b-1} - 1)$

We get

$$P_6 := P_{6.1} + P_{6.2} = 4t \cdot p^3q \cdot (p^2q^2)^{b-1} \cdot (6^{b-1} - 3 \cdot 4^{b-1} - 3^{b-1} + 6 \cdot 2^{b-1} - 3).$$

Case 7: All b-vertices have degree 2.

Case 7.1: All classes of degree-2-vertices are non-empty.
$P_{7.1} := (p^2q^2)^b \cdot (6^b - 6 \cdot 5^b + 15 \cdot 4^b - 20 \cdot 3^b + 15 \cdot 2^b - 6)$

Case 7.2: Exactly one class is empty.
$P_{7.2} := (p^2q^2)^b \cdot 6 \cdot (5^b - 5 \cdot 4^b + 10 \cdot 3^b - 10 \cdot 2^b + 5)$

Case 7.3: Exactly two classes are empty, which have no a-vertex in common.
$P_{7.3} := (p^2q^2)^b \cdot 3 \cdot (4^b - 4 \cdot 3^b + 6 \cdot 2^b - 4)$

Therefore we get

$$P_7 := P_{7.1} + P_{7.2} + P_{7.3} = (p^2q^2)^b \cdot (6^b - 12 \cdot 4^b + 28 \cdot 3^b - 27 \cdot 2^b + 12).$$

Altogether, we get

$$R^2(K_{4,t}) = P_1 + P_{2.1} + P_{2.2} + P_{2.3} + P_3 + P_{4.1} + P_{4.2} + P_{4.3} + P_5 + P_6 + P_7.$$

The theorem follows by applying the Binomial theorem to the sums over k. □

Conclusion:

> By distinction of possible vertex degrees, it is possible to get an explicit formula for every fixed a. Unfortunately, the number of distinct success cases, which need to be considered, becomes vast already for relatively small a. Hence, in the following section a recursive formula for arbitrary a will be developed.

4.4.3 A recurrence relation for the biconnected reliability polynomial of $K_{a,b}$

We consider the event that after edge failure the graph $K_{a,b} = (A \cup B, E)$ remains connected. Let v be an arbitrary chosen, fixed a-vertex. Let G be a random spanning subgraph of $K_{a,b}$ resulting after edge failure. Depending on the connectivity towards v, we partition our vertex set in the following sets (see Figure 4.6 for an example):

- Let F denote the set of bridges incident to v. Let I denote the connected component of v in $G-F$. We define $S := V(G) \setminus V(I)$.
- Let K denote the set of all vertices, which form a block with v (if v is contained in more then one block, one block is arbitrarily chosen for K).
- Let H be the connected component containing v in $G-(K \setminus \{v\})$. We define $T := V(H) \setminus (S \cup \{v\})$ and $L := V(G) \setminus (V(H) \cup K)$.

Every vertex of L is connected to v via a cutpoint in $K \setminus \{v\}$. Depending whether this cutpoint is an a- or b- vertex, we divide L into L^a and L^b.
We further distinguish, whether the vertices belong to a or to b:
$X_a := X \cap A,\ X_b := X \cap B;\ X \in \{S, K, T, L^a, L^b\}$
With the corresponding lower-case letters we will denote the size of those vertex sets.
Clearly it holds

$$\begin{aligned} s_a + k_a + t_a + l_a^a + l_a^b &= a \quad \text{and} \\ s_b + k_b + t_b + l_b^a + l_b^b &= b. \end{aligned}$$

We will now develop formulas for the probability of the different vertex sets.

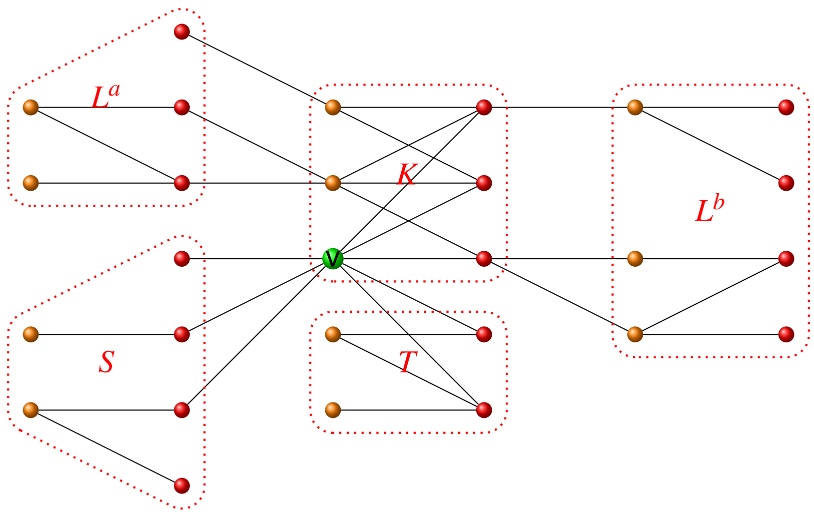

Figure 4.6: Vertex partitioning depending on connectivity to fixed vertex v after edge failure

Lemma 4.20 *Let s,t be non-negative integers. Consider the complete bipartite graph $K_{s+1,t}$ with $A = \{a_1, \ldots, a_s, v\}$ under edge failure. Let $p_1(a,b)$ denote the probability that the remaining graph is connected and all edges towards v are bridges. Then it holds*

$$p_1(s,t) = q^{s\cdot t} \cdot \sum_{\tau \vdash t} \left\{ {t \atop \tau} \right\} \cdot (1-q)^{|\tau|} q^{t-|\tau|} \cdot \sum_{\sigma \vDash (s,|\tau|,0)} \binom{s}{\sigma} \cdot \prod_{i=1}^{|\tau|} R(K_{\sigma_i,\tau_i}) \cdot \tau_i \cdot q^{-\sigma_i \tau_i}.$$

Remark 4.21 We will use Lemma 4.20 to account for the vertex set S consisting of vertices connected to v via bridges.

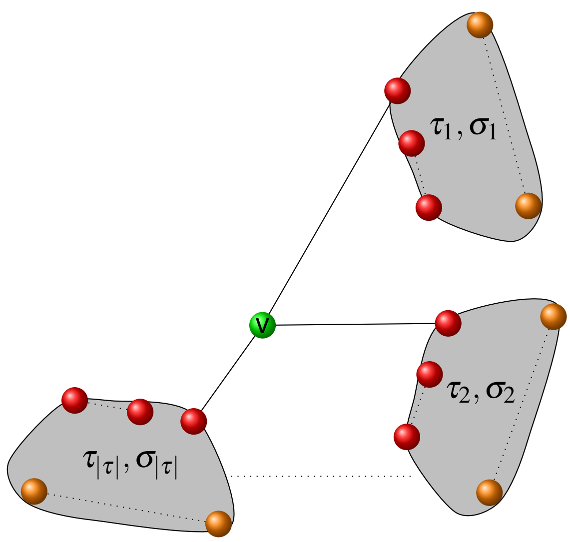

Figure 4.7: All edges adjacent to v are bridges to the parts of size τ_i, σ_i.

Proof: Denote the resulting graph after edge failure with G. Let $|\tau|$ denote the number of components of $G-v$ and τ_i and σ_i denote the number of b- and a- vertices respectively in the ith connected component (see Figure 4.7 for the structure of G). Note, that each component must contain a b-vertex ($\tau_i \geq 1$) while it is possible that a component consists solely of one b-vertex and no a-vertices ($\sigma_i \geq 0$). Given the size of the components, we have $\left\{ \begin{matrix} t \\ \tau \end{matrix} \right\}$ ways to assign the b-vertices to the components and afterwards we can choose the a-vertices in $\binom{s}{\sigma}$ ways. For every component the following properties need to hold:

- The component must be connected intrinsic.
- All edges to the other components fail.
- One edge to v is intact, the others fail.

Since these properties describe disjoint edge sets, the probabilities are independent and can be multiplied. The probability for that is:

$$\prod_{i=1}^{|\tau|} \underbrace{R(K_{\sigma_i,\tau_i})}_{\text{connected}} \cdot \underbrace{\tau_i \cdot (1-q) q^{\tau_i - 1}}_{\text{one edge to} v} \cdot \underbrace{q^{\sigma_i \cdot (t-\tau_i)}}_{\substack{\text{edges between different} \\ \text{components fail}}}$$

We sum over all possible sizes for τ_i and σ_i.
By using $t = \sum_i \tau_i$, $\sum_i \sigma_i (t-\tau_i) = st - \sum_i \sigma_i \tau_i$ the result follows. □

Lemma 4.22 *Consider the complete bipartite graph $K_{s+1,t}$ with $A = \{a_1, \ldots, a_s, v\}$ under edge failure. Let $p_2(b,s,t)$ denote the probability that the remaining graph is connected and v is contained in exactly b blocks and no edges incident to v are bridges. Further let $p_2(s,t) := \sum_b \frac{p_2(b,s,t)}{b+1}$. Then it holds*

$$p_2(s,t) = q^{s \cdot t} \cdot \sum_{\sigma \vdash s} \left\{ \begin{matrix} s \\ \sigma \end{matrix} \right\} \frac{1}{|\sigma|+1} \sum_{\tau \vDash (t,|\sigma|,2)} \binom{t}{\tau}$$
$$\cdot \prod_{i=1}^{|\sigma|} R(K_{\sigma_i,\tau_i}) \cdot (1 - q^{\tau_i} - \tau_i (1-q) q^{\tau_i - 1}) \cdot q^{-\sigma_i \cdot \tau_i}.$$

Remark 4.23 We will use this lemma to describe the connectivity of the vertex set T to v. The term $\frac{1}{b+1}$ accounts for the different choices of K out of $b+1$ blocks.

Proof: Denote the remaining graph after edge failure with G. Since no edges incident to v are bridges, $G-v$ decomposes in b connected components (where the number of vertices is denoted by σ_i and τ_i), all of them having at least two intact edges towards v in G. Since v is adjacent only to b-vertices and these b-vertices must be connected via an a-vertex in the component, it follows $\sigma_i \geq 1$ and $\tau_i \geq 2$ for all i. We have $\left\{ {s \atop \sigma} \right\}$ ways to assign the a-vertices to the components and $\binom{t}{\tau}$ ways to choose the b-vertices. For every component the following properties need to hold:

- The components are connected intrinsic.
- All edges between different components fail.
- At least 2 edges to v must remain intact.

The probability for that is

$$\prod_{i=1}^{b} \underbrace{R(K_{\sigma_i,\tau_i})}_{\text{connected}} \cdot \underbrace{\left(1-q^{\tau_i}-\tau_i(1-q)q^{\tau_i-1}\right)}_{\text{at least 2 edges to v}} \cdot \underbrace{q^{\sigma_i(t-\tau_i)}}_{\substack{\text{edges between different} \\ \text{components fail}}} .$$

It remains to sum over all sizes of τ_i and σ_i. By using $\sum_i \sigma_i = s$, we get

$$p_2(b,s,t) = \sum_{\substack{\sigma \vdash s \\ |\sigma|=b}} \left\{ {s \atop \sigma} \right\} \cdot \sum_{\tau \vDash (t,|\sigma|,2)} \binom{t}{\tau} \cdot q^{s \cdot t} \cdot \prod_{i=1}^{b} R(K_{\sigma_i,\tau_i}) \cdot (1-q^{\tau_i}-\tau_i(1-q)q^{\tau_i-1}) \cdot q^{-\sigma_i \cdot \tau_i}.$$

The result then follows immidiatly from $p_2(s,t) := \sum_b \frac{p_2(b,s,t)}{b+1}$. □

Lemma 4.24 *Consider the complete bipartite graph $K_{s,t+k}$ with $B = \{\underbrace{t_1,\ldots,t_t}_{T},\underbrace{k_1,\ldots,k_k}_{K}\}$. Let $p_3(k,s,t)$ describe the probability that after edge failure the remaining graph decomposes into exactly k connected components and no two vertices of K are in the same component. Then it holds:*

$$p_3(k,s,t) = \sum_{\sigma \vdash s} \left\{ {s \atop \sigma} \right\} k^{\underline{|\sigma|}} \cdot \sum_{\tau \vDash (t,|\sigma|,0)} \binom{t}{\tau} \cdot q^{s \cdot t + s \cdot (k-1)} \cdot \prod_{i=1}^{|\sigma|} R(K_{\sigma_i,\tau_i+1}) \cdot q^{-\sigma_i \cdot \tau_i}.$$

Remark 4.25 We will use this to describe the vertex sets L_s and L_t which are connected via an articulation in $K \setminus \{v\}$.

Proof: Let $|\sigma|$ denote the number of vertices of K which are not isolated after edge failure. Let σ_i and τ_i denote the number of vertices of A and T respectively connected to the ith non-isolated vertex of K. Hence, it holds: $\sigma_i \geq 1$, $\tau_i \geq 0$. We have $\left\{ {s \atop \sigma} \right\}$ ways to choose the assignment of A towards the connected components. Afterwards we have $k^{\underline{|\sigma|}}$ ways to assign the corresponding non-isolated vertices of K and $\binom{t}{\tau}$ ways to choose the assignment of the vertices of T. The components must be connected and all edges towards other components need to fail. We get

$$\prod_{i=1}^{|\sigma|} R(K_{\sigma_i,\tau_i+1}) \cdot q^{\sigma_i\cdot(k-1)} \cdot q^{\sigma_i\cdot(t-\tau_i)}.$$

It remains to sum over all sizes of σ_i and τ_i. □

Theorem 4.26 *For the complete bipartite graph $K_{a,b}$, $a,b \geq 2$ holds*

$$\begin{aligned} R(K_{a,b}) =& p_1(a-1,b) \\ &+ \sum_{k_a=2}^{a}\sum_{k_b=2}^{b}\binom{a-1}{k_a-1}\binom{b}{k_b} R^2(K_{k_a,k_b}) \\ &\cdot \sum_{t_a=0}^{a-k_a}\sum_{t_b=0}^{b-k_b}\binom{a-k_a}{t_a}\binom{b-k_b}{t_b} p_2(t_a,t_b) \\ &\cdot \sum_{l_a^a=0}^{a-k_a-t_a}\sum_{l_b^a=0}^{b-k_b-t_b}\binom{a-k_a-t_a}{l_a^a}\binom{b-k_b-t_b}{l_b^a} p_3(k_a-1,l_b^a,l_a^a) \\ &\cdot \sum_{l_a^b=0}^{a-k_a-t_a-l_a^a}\sum_{l_b^b=0}^{b-k_b-t_b-l_b^a}\binom{a-k_a-t_a-l_a^a}{l_a^b}\binom{b-k_b-t_b-l_b^a}{l_b^b} \\ &\cdot p_3(k_b,l_a^b,l_b^b)\cdot p_1(s_a,s_b)\cdot q^c \end{aligned}$$

with $s_a = a-k_a-t_a-l_a^a-l_a^b$; $s_b = b-k_b-t_b-l_b^a-l_b^b$ *and*

$$\begin{aligned} c =& s_a\cdot(b-s_b)+s_b\cdot(a-1-s_a)+t_a\cdot(b-t_b-s_b)+t_b\cdot(a-1-t_a-s_a) \\ &+ l_a^a\cdot l_b^b + l_b^a\cdot l_a^b + l_b^a + l_a^a\cdot k_b + l_b^b\cdot k_a. \end{aligned}$$

Proof: We consider the event that after edge failure the remaining graph remains connected. We consider a fixed a-vertex v. If v is not within a block, all edges incident to v must be bridges and the probability can be described by $p_1(a-1,b)$. Otherwise, we partition the vertex set depending on the connectivity towards v in the vertex sets S,K,T,L^b,L^a described in the beginning of this section. Given the size of the vertex sets, we have $\binom{a-1}{k_a-1,t_a,l_a^a,l_a^b,s_a}$ choices for the a-vertices without v into the different sets and $\binom{b}{k_b,t_b,l_b^a,l_b^b,s_b}$ choices for the b-vertices. The connectivity of the vertex set K is then described by $R^2(K_{k_a,k_b})$, the connectivity of S is described by $p_1(s_a,s_b)$, the connectivity of T is described by $p_2(t_a,t_b)$ and the connectivity L_a and L_b is described by $p_3(k_a-1,l_b^a,l_a^a)$ and $p_3(k_b,l_a^b,l_b^b)$ respectively. It remains to consider the necessary edge failures between the different sets with q^c. □

Remark 4.27 We can easily derive a recurrence relation from Theorem 4.26 by rearranging terms.

4.4.4 Two-edge connected reliability polynomial of $K_{a,b}$

For the two-edge connected reliability polynomial of the complete bipartite graph we consider the event that the remaining graph after edge failure is connected. We consider a fixed a-vertex v and partition the vertex set depending on the connectivity to v.

Theorem 4.28 *For the two-edge connected reliability polynomial of $K_{a,b}$ holds*

$$\begin{aligned} R(K_{a,b}) =& p_1(a-1,b) \\ &+ \sum_{k_a=2}^{a} \sum_{k_b=2}^{b} \binom{a-1}{k_a-1} \binom{b}{k_b} R_{2-ec}(K_{k_a,k_b}) \\ &\cdot \sum_{l_a=0}^{a-k_a} \binom{a-k_a}{l_a} (k_b(1-q)q^{k_b-1})^{l_a} \sum_{l_b=0}^{b-k_b} \binom{b-k_b}{l_b} (k_a(1-q)q^{k_a-1})^{l_b} \\ &\cdot p_2(k_a,k_b,a-k_a-l_a,b-k_b-l_b) \cdot q^c \end{aligned}$$

with $c = l_a(b-k_b-l_b) + l_b(a-k_a)$, p_1 *defined in Lemma 4.20 and*

$$\begin{aligned} p_2(k_a,k_b,s_a,s_b) :=& q^{k_b s_a + k_a s_b + s_a s_b} \sum_{\sigma \vdash s_a} \sum_{\tau \vDash (s_b,|\sigma|)} \left\{ {s_a \atop \sigma} \right\} \binom{s_b}{\tau} \cdot (1-q)^{|\sigma|} q^{-|\sigma|} \\ &\cdot \prod_{i=1}^{|\sigma|} R(K_{\sigma_i,\tau_i}) \cdot (\sigma_i k_b + \tau_i k_a) \cdot q^{-\sigma_i \cdot \tau_i}. \end{aligned}$$

Proof: We consider the event that the remaining graph after edge failure is connected. Let v be a fixed a-vertex. Then the following events might occur:

- The vertex v is not in a two-edge connected component.
  This occurs if and only if v is not part of a block, so the probability for this event can be described exactly as for the biconnected reliability: $p_1(a-1,b)$ as described in Lemma 4.20.
- The vertex v is in a two-edge connected component.
  Then we distinguish the following vertex sets (see Figure 4.8 for an example):
  - We denote by K the vertices of this two-edge connected component.
  - Let $H := G - K$. Then let L denote the set of isolated vertices of H and S denote the vertices in components of size ≥ 2 (which therefore have at least one a- and b-vertex).

Let K_a and K_b denote the vertices of K which are a-vertices and b-vertices, respectively.

Analogously, define L_a, L_b, S_a and S_b. Let the corresponding lower case letters denote the sizes of these sets. Then it clearly holds

$$\begin{aligned} a = &\quad k_a + l_a + s_a \quad \text{and} \\ b = &\quad k_b + l_b + s_b. \end{aligned}$$

We consider a fixed assignment of the vertices towards the sets K, L and S. Then the probability for K to be a two-edge connected component is $R_{2-ec}(K_{k_a,k_b})$.
Since the vertices of L are isolated in H, every vertex of L_a (L_b) must be of degree 1 and the operating edge is incident to a vertex of K_b (K_a). The probability for this edge failure towards K is $\left(k_b \cdot (1-q) \cdot q^{k_b-1}\right)^{l_a}$ and
$\left(k_a \cdot (1-q) \cdot q^{k_a-1}\right)^{l_b}$ respectively.
For the set S now consider a certain assignment of the vertices towards the connected components in H. Let $|\sigma|$ denote the number of those components and $\sigma_1, \ldots, \sigma_{|\sigma|}$ the sizes of the a-vertex sets of the components. Let $\tau_1, \ldots, \tau_{|\sigma|}$ be the sizes of the corresponding b-vertex sets. For every component, the following properties need to hold: Exactly one edge towards K needs to be operating, the components must be connected intrinsically and all edges between different components need to fail. The probability for this is

$$\underbrace{(1-q)^{|\sigma|} q^{k_b s_a + k_a s_b - |\sigma|}}_{\substack{\text{every component has exactly} \\ \text{one operating edge to } K}} \cdot \prod_{i=1}^{|\sigma|} \underbrace{R(K_{\sigma_i,\tau_i})}_{\substack{\text{intrinsically} \\ \text{connected}}} \cdot \underbrace{(\sigma_i k_b + \tau_i k_a)}_{\substack{\text{choices for} \\ \text{operating edge to } K}} \cdot \underbrace{q^{\sigma_i (s_b - \tau_i)}}_{\substack{\text{all edges to other} \\ \text{components fail}}} .$$

By considering all assignments of S towards the components, we get the term described by $p_2(k_a, k_b, s_a, s_b)$.
All edges between S and L and all edges in L need to fail, which results in the term q^c with $c = l_a(b - k_b - l_b) + l_b(a - k_a - l_a) + l_a \cdot l_b$.
It remains to sum over all assignments to K, L and S and combining the corresponding subresults. □

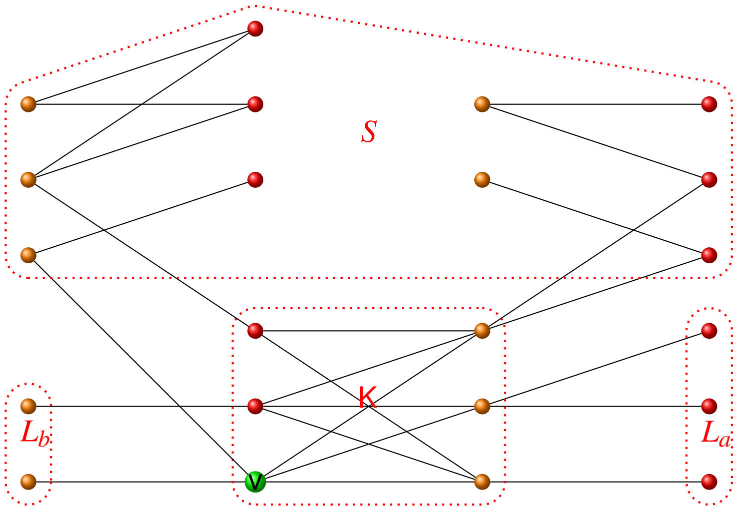

Figure 4.8: Vertex partitioning depending on connectivity to fixed vertex v after edge failure

5 Minimally biconnected graphs, essential and irrelevant edges

In practical applications analyzing which edges of a graph are irrelevant and essential may help to reduce the size of the graph.

Definition 5.1 A graph G is minimally biconnected, if G is biconnected but for all $e \in E$, $G - e$ is not biconnected. That means that every edge of G is essential.

Definition 5.2 The block-cutvertex graph of the connected graph G, denoted by $bc(G)$, is the graph whose vertices are the blocks (including K_2 as block) and cut vertices of G. The edges of $bc(G)$ join cut vertices with those blocks to which they belong.

The following theorems were obtained independently by Dirac [Dir67] and Plummer [Plu68]. We will present the proofs as given in [Bol04].

Theorem 5.3 *Let $G = (V,E)$ be a biconnected graph and $e = \{x,y\} \in E$ be an edge of G. Let $H := G - e$ be the graph resulting from G after deletion of e. Then the following two assertions are equivalent.*

1. *The edge e is an essential edge of G.*
2. *The block-cutvertex graph of H, $bc(H)$, has a non-trivial $x-y$-path, x belongs to the initial block and y belongs to the terminal block.*

Proof: It is clear that 2 implies 1. So we suppose that 1 holds and intend to prove 2.
It is immiadiate that H is connected (since G is 2-connected) and $bc(H)$ is a non-trivial tree (since H is connected and not biconnected). If L is any graph with $u, v \in V(L)$ and $M = L + \{u,v\}$ then $M - u = L - u$. So if we add an edge to an articulation u, then u remains an articulation. Consequently x and y are no articulations of H (since G is biconnected). To complete the proof it suffices to show that the tree $bc(H)$ does not contain an end vertex C (a block of H) such that neither x nor y is a vertex of C. Let c be the unique cutvertex of H in C. Then the addition of e to H does not join a vertex of $C - c$ to a vertex $H - C$, i.e. c is a cutvertex of H as well, contradicting the assumption

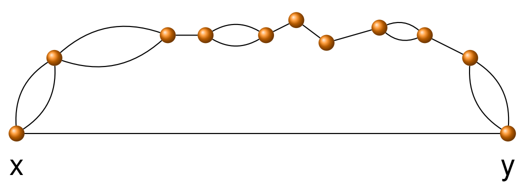

Figure 5.1: Structure of G in Theorem 5.3

that G is biconnected. □

Corollary 5.4 *Let G be a biconnected graph. An edge $e = \{u, v\}$ is an essential edge of G if and only if no cycle of $G - e$ contains both x and y, i.e. e is not a diagonal of a cycle in G.*
In particular, a biconnected graph is minimally biconnected if and only if no cycle has a diagonal.

Proof: If no cycle of $G - e$ contains both x and y then $G - e$ is not biconnected so e is an essential edge. The converse implication follows from Theorem 5.3. □

Corollary 5.5 *Every biconnected subgraph of a minimally biconnected graph is minimally biconnected.*

Corollary 5.6 *Let G be a biconnected graph of order at least 4. Then G does not contain a triangle formed by essential edges.*

Proof: Suppose $x, y, z \in V$ form such a triangle and $u \in V \setminus \{x, y, z\}$. As G is biconnected, it follows from Menger's Theorem that there exist two independent paths from u to $\{x, y, z\}$, say a $u - x$-path P_1 and a $u - y$-path P_2, with $z \notin P_1 \cup P_2$. Then $\{x, y\}$ is a diagonal of the cycle uP_1xzyP_2u, contradicting Corollary 5.4. □

Further properties of minimally biconnected graphs can be found in [Bol04].

In Theorem 3.1 and Theorem 3.2 we characterized loops and edges joining separators of cardinality two as irrelevant. Further irrelevant edges exist (see Figure 5.2 for an example), while a characterization of those remains unknown. Those irrelevant edges were identified via essential edges (see the following paragraph). This motivated us to the following conjectures:

Conjecture 5.7 *Let G be a simple two-connected graph with minimal degree $\delta \geq 3$. Then $e = \{u, v\} \in E$ is irrelevant if and only if $\{u, v\}$ is a separator.*

Conjecture 5.8 *Let G be a simple two-connected graph without essential edges. Then $e = \{u, v\} \in E$ is irrelevant if and only if $\{u, v\}$ is a separator.*

Conjecture 5.9 *Let G be a simple three-connected graph. Then G does not contain irrelevant edges.*

It is obvious that Conjecture 5.7 implies Conjecture 5.8 and Conjecture 5.8 implies Conjecture 5.9. Yet, no proof or counterexample for any of the conjectures was found.

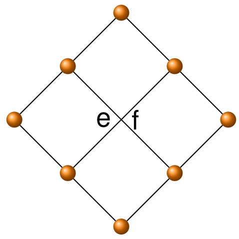

Figure 5.2: Graph with irrelevant edges e and f which do not join separators of cardinality two

For practical applications we can delete all edges e with $R^2(G) = R^2(G-e)$. All irrelevant edges fulfill this property. Further, edges having probability one (by default or after applying reductions/decomposition formula), may result in other edges becoming dispensable:

Theorem 5.10 *Let $G = (V,E)$ be a probabilistic graph. Let $F \subseteq E$ be a circle of fail-proof edges and $W \subseteq V$ the vertices incident to F. Then for all edges $e = \{u,v\} \in W^2 \setminus F$ holds:*

$$R_K^2(G) = R_K^2(G-e)$$

Further, if W is a separator of G, then it holds:

$$R^2(G) = R^2(G_1) \cdot R^2(G_2)$$

Proof: The biconnected reliability can be expressed via success sets as described in Equation 2.1. All success sets A with $F \not\subseteq A$ account with zero in this sum due to the term $(1-p_f)$ for some $f \in F$. Hence we could limit the set of success sets to those containing F and considering minimal success sets of those. Then e is in no minimal success set, since two disjoint paths from u to v can always be obtained via F.
If W is a separator, then $x \in V_1$ and $y \in V_2$ can only be in the same block if there exist two disjoint paths traversing through W. Denote these paths as $xP_1w_1Q_1y$ and $xP_2w_2Q_2y$. Then $xP_1w_1F'w_2P_2x$ with $F' \subseteq F$ is a cycle in G_1 and hence x, w_1 and w_2 are in the same block in G_1. The same holds for y in G_2. Therefore all vertices of V are in the same block in G if all vertices of V_1 are in the same block in G_1 and all vertices of V_2 are in the same block of V_2. If all vertices are in the same block in G_1 and G_2, they are in the same block in G and hence equality holds. □

6 Summary and Prospect

Since the underlying problem is NP-hard, algorithms for general graphs are likely to remain exponentiell. For certain graph classes however, algorithms were found which outperform complete enumeration of all states. Further, we presented some reductions which can reduce the size of a considered graph or resulting graphs during decomposition and extended some of those to other reliability measures. For some applications it may be sufficient to know, whether the reliability surpasses a critical value. Hence, the analysis of lower and upper bounds of the biconnected reliability might become a topic of interest in following studies. Additionally we analyzed undirected networks. For practical applications some connections allow only one-sided communication. The analysis of directed networks might be appropriate. Especially the analysis of acyclic graphs and the conversion of an undirected to a directed network with the same reliability seem feasible tasks. Further analysis considering k-connectivity of probabilistic graphs or transfer capacities may be part of upcoming studies as well.

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

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Mittweida, 3.08.2015